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Characterization of Nongaussian Atmospheric Turbulence for Prediction of Aircraft Response Statistics

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Characterization of Nongaussian Atmospheric Turbulence for Prediction of Aircraft Response Statistics

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LIST OF SYMBOLS

A	constant
A'	constant
a_n	expansion coefficient of instantaneous spectrum
B	constant
b_n	expansion coefficient
C	Euler's constant, 0.577215665...
Cov [...]	covariance function
$E\{...\}$	mathematical expectation
$E\{...\vert...\}$	conditional expectation
$F(\xi)$	Arcsin ξ
${}_2F_1$	hypergeometric function
f	frequency
g	"dummy" frequency variable
H	complex frequency response function
h	unit-impulse response function of aircraft response
i	$\sqrt{-1}$
K	constant
k	index
L_z	integral scale of turbulence component z
ln	natural logarithm
M	index

$M^{(k)}$	kth order moment
$m_h^{(k)}(f)$	kth order power-moment spectrum of impulse response function
N	index
n	index
$N_+^{(k)}(y\|\sigma_f^2)$	kth derivative of $N_+(y\|\sigma_f{}^2)$ with respect to σ_f^2
$N_+(y)$	mean up-crossing exceedance rate of "level" y
$\binom{n}{k}$	binomial coefficient
p	probability density function
$p^{(k)}(y\|\sigma_f^2)$	kth derivative of $p(y\|\sigma_f^2)$ with respect to σ_f^2
$Q^{(k)}$	kth order correction term to Gaussian up-crossing rate
R	autocorrelation function
$R^{(k)}$	kth derivative of R
R_0	autocorrelation function of hard clipped version of W_h
t	time
u	"dummy" variable, natural logarithm of z_h^2
$U^{(k)}$	kth order correction term to Gaussian pdf
V	aircraft speed, abbreviation for σ_f^2
v	natural logarithm of σ_f^2
Var	variance
w	turbulence velocity

w_f	"fast"turbulence component
w_h	high-pass filtered version of turbulence component w
w_s	"slow" turbulence component
x	"dummy" variable, spatial variable
y	aircraft response
z	Gaussian factor of "fast" turbulence component
z_h	high-pass filtered version of turbulence component z

LIST OF SYMBOLS *(Cont.)*

Greek Symbols:

α_{nk}	expansion coefficient
Γ	gamma function
γ	constant
$\gamma^{(2)}$	coefficient of excess
δ	Dirac delta function
$\theta_h(f)$	phase of complex frequency response function
λ	constant
$\mu^{(k)}$	kth order central moment
ξ	"dummy" variable, spatial lag variable
π	3.1415926535 ...
ρ	correlation coefficient
σ	standard deviation
σ_f	(stochastic) standard deviation of "fast" turbulence component
σ_y	standard deviation of aircraft response
τ	temporal lag
$\tau_h(f)$	group delay of aircraft impulse response function
Φ	power spectrum
ϕ	autocorrelation function
$\Phi^{(n)}$	nth derivative of power spectrum

INTRODUCTION AND SUMMARY

In the 1950's, Press and others developed expressions for the mean exceedance rate $N(y)$ of an arbitrary aircraft response variable through a generic response level y. In deriving these expressions, they assumed that the aircraft response is a locally stationary, locally Gaussian random process; the results were based on Rice's famous formula for the mean rate of threshold crossings of a stationary Gaussian process. Modeling the turbulence as a locally stationary, locally Gaussian random process – generally with a Dryden spectrum – permitted the turbulence to be characterized by its integral scale and the probability density function of its standard deviation. The fact that the standard deviation of the turbulence was treated as a slowly fluctuating random variable permitted the mathematical representation of the turbulence to model the patchlike character of real turbulence.

Measurements recently obtained in a project being carried out at the NASA Langley Research Center have demonstrated the existence of a low-frequency (large wavenumber) component in many turbulence recordings, where this "slow" turbulence component $w_s(t)$ appears to fluctuate independently of the patch-like character associated with the turbulence model used by Press and more recent investigators. The addition of this large wavenumber component suggests that turbulence velocity records $w(t)$ be modeled by a three-component random process†

$$w(t) = w_s(t) + w_f(t) = w_s(t) + \sigma_f(t)\, z(t), \qquad (1.1,\ 2.3)$$

where

$$E\{z\} = 0,\ E\{z^2\} = 1 \qquad (1.2,\ 2.5)$$

and where the standard deviation of $\sigma_f(t)$ of the "fast" turbulence component $w_f(t)$ satisfies $\sigma_f(t) \geq 0$. In the work reported herein, we have assumed that the random processes $\{w_s(t)\}$, $\{\sigma_f(t)\}$,

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†Most of the equations in this introductory section have two numbers. The first number designates the order of appearance of the equation in the present section; the second designates the number associated with the same equation, as it appears later in the report where the material is treated in detail.

and $\{z(t)\}$ are all stationary and mutually independent. Also, we have assumed that $z(t)$ is a Gaussian process and, in some places, that $w_s(t)$ also is Gaussian. Further discussion of this model is provided in Sec. 2 of the report.

One of the central tasks of the present work has been to determine the conditions that must be satisfied for validity of the locally stationary, locally Gaussian response approximation. To accomplish this task, we have used the concept of the turbulence process $\{w(t)\}$ conditioned on the behavior of the process $\sigma_f(t)$. This conditioning operation is equivalent to dealing with the stochastic behavior of $\{w(t)\}$ while assuming that the function $\sigma_f(t)$ is completely specified. Since the processes $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$ are assumed to be mutually independent, this conditioning operation presents no conceptual difficulties. Thus, we are able to form expressions for the conditional instantaneous autocorrelation function and its Fourier transform, the conditional instantaneous spectrum of the turbulence process $\{w(t)\}$, given that the function $\sigma_f(t)$ is specified. These expressions are derived in Sec. 3.1.

Although the process $\{w(t)\}$ is stationary, the process $\{w(t)\}$ conditioned on $\sigma_f(t)$ is, in general, nonstationary. However, in an earlier study that dealt with the effects of nonstationary behavior on the spectra of atmospheric turbulence, a series expansion of the instantaneous spectrum was developed for studying the effects of the time variations of $\sigma_f(t)$ on the instantaneous spectrum. Some results of this earlier study, relevant to the present work, are reviewed in Sec. 3.2. The first term in this series expansion, when applied to the "fast" component $w_f(t)$ in Eq. (1.1) and conditioned on the function $\sigma_f(t)$, is the usual locally stationary spectrum approximation

$$\Phi_{w_f}(f,t|\sigma_f) \approx \sigma_f^2(t)\ \Phi_z(f), \tag{1.3, 3.11}$$

where $\Phi_z(f)$ is the power spectrum of the stationary process $\{z(t)\}$ of Eq. (1.1). Thus, investigation of the correction terms provided by the series expansion have enabled us to formulate conditions for validity of the locally stationary response approximation.

First, it was necessary to derive a series expansion for the instantaneous spectrum of the response process, also conditioned on the behavior of $\sigma_f(t)$. To obtain this expansion, we have used the input-response relationship for the instantaneous spectrum derived in our earlier report. The conditioned instantaneous spectrum of the aircraft response process $\{y(t)\}$ is given by Eq. (3.29) of the present report. When the series

expansion for $\Phi_w(f,t|\sigma_f)$ is combined with this input-response relationship, we obtain the series expansion for the conditioned instantaneous response spectrum given by Eq. (3.37) or (3.38) and derived in Sec. 3.4. The leading terms in this response representation are

$$\Phi_y(f,t|\sigma_f) = [\Phi_{w_s}(f)+\sigma_f^2(t)\Phi_z(f)]\ |H(f)|^2, \qquad (1.4,\ 3.40)$$

where $\Phi_{w_s}(f)$ is the power spectrum of the "slow" component $w_s(t)$ in the turbulence model of Eq. (1.1) and $H(f)$ is the aircraft complex frequency-response function. Equation (1.4) is the obvious locally stationary response approximation that could have been written directly from Eq. (1.1).

Examination of the appropriate correction terms in Eq. (3.38) to the locally stationary response approximation given by Eq. (1.4) has enabled us to write conditions for validity of the locally stationary response approximation of Eq. (1.4) in Sec. 3.5. Three conditions, Eqs. (3.41), (3.43), and (3.46) are given. Equation (3.41) expressed the local stationarity requirement for the turbulence $w(t)$ of Eq. (1.1), whereas Eqs. (3.43) and (3.46) express the local stationarity requirements for the aircraft response $y(t)$, assuming that the requirement of Eq. (3.41) is satisfied.

The local stationarity requirements of Eqs. (3.41), (3.43), and (3.46) are expressed in terms of (derivatives of the logarithm of) the function $\sigma_f^2(t)$, which is still assumed to be a specified function at this juncture. Before discussing these requirements and formulating them in terms of stochastic metrics of $\sigma_f^2(t)$, we derive expressions for the aircraft-response exceedance rate $N_+(y)$ and the first-order probability density function $p(y)$. These expressions are derived in Sec. 4, where we assume validity of the locally stationary response approximation given by Eq. (1.4).

In Sec. 4.1, it is shown that if the process $\{z(t)\}$ in the model of Eq. (1.1) is Gaussian then the process $\{w_f(t)|\sigma_f\}$, conditioned on the process $\sigma_f(t)$, also is Gaussian - this result being independent of locally stationary requirements. However, if probability density functions of $w_f(t)$ are generated by time averages, then local stationarity is required for the "fast" process $\{w_f(t)\}$ to be considered locally Gaussian.

In Sec. 4.2, we derive the expression for response exceedance rates (with positive slopes) given by

$$N_+(y) = \int_0^\infty N_+(y|\sigma_f^2)\ p(\sigma_f^2)d\sigma_f^2\ , \qquad (\ .5,\ 4.8)$$

where $p(\sigma_f^2)$ is the probability density of the square of the process $\sigma_f(t)$ – $\sigma_f^2(t)$, therefore, being the local variance of the fast turbulence component $w_f(t)$ – and where $N_+(y|\sigma_f^2)$ is the expected local up-crossing rate through the threshold y of the response process, given that the local value of $\sigma_f^2(t)$ is specified and assuming the locally stationary response approximation of Eq. (1.4) to be justified. The expression for $N_+(y|\sigma_f^2)$ is given by Eq. (4.22). To evaluate $N_+(y|\sigma_f^2)$ we require spectra $\Phi_z(f)$ and $\Phi_{w_s}(f)$ of the turbulence components $z(t)$ and $w_s(t)$, as well as the magnitude of the aircraft frequency-response function.

For cases where variations in $\sigma_f^2(t)$ are small relative to the expected value of $\sigma_f^2(t)$, a useful series expansion for $N_+(y)$ has been derived in Sec. 4.3. This series expansion is of the form

$$N_+(y) = \sum_{k=0}^{\infty} \frac{1}{k!}\ N_+^{(k)}(y|\overline{\sigma_f^2})\mu_{\sigma_f^2}^{(k)}\ , \qquad (1.6,\ 4.28)$$

where we have defined

$$N_+^{(k)}(y|\overline{\sigma_f^2}) \triangleq \left.\frac{d^k N_+(y|\sigma_f^2)}{d(\sigma_f^2)^k}\right|_{\sigma_f^2 = \overline{\sigma_f^2}}\ , \qquad (1.7,\ 4.26)$$

where

$$\overline{\sigma_f^2} = E\{\sigma_f^2\} \qquad (1.8,\ 4.24)$$

is the expected value of σ_f^2 and where

$$\mu_{\sigma_f^2}^{(k)} \triangleq \int_0^\infty (\sigma_f^2-\overline{\sigma_f^2})^k\ p(\sigma_f^2)\ d(\sigma_f^2),\ k = 1,2,\cdots \qquad (1.9,\ 4.29)$$

are the central moments of $\sigma_f^2(t)$. Of particular interest is the two-term approximation of Eq. (1.6) given by

$$N_+(y) \simeq N_+(y|\overline{\sigma_f^2}) + \frac{1}{2}\,\mu^{(2)}_{\sigma_f^2}\; N_+^{(2)}(y|\overline{\sigma_f^2}) \qquad (1.10,\ 4.31)$$

$$= N_+(y|\overline{\sigma_f^2})\ [1+\mu^{(2)}_{\sigma_f^2} Q^{(2)}(y|\overline{\sigma_f^2})], \qquad (1.11,\ 4.36)$$

where $N_+^{(2)}(y|\overline{\sigma_f^2})$ has been written in the second line as

$$N_+^{(2)}(y|\overline{\sigma_f^2}) = 2N_+(y|\overline{\sigma_f^2})\ Q^{(2)}(y|\overline{\sigma_f^2}). \qquad (1.12,\ 4.33)$$

From Eq. (1.10), it is evident that the first term $N_+(y|\overline{\sigma_f^2})$ on the right-hand side is the exceedance rate one obtains by assuming that $y(t)$ is a stationary Gaussian process.* Thus, the second term in Eq. (1.10) is a correction term that modifies the Gaussian approximation $N_+(y|\overline{\sigma_f^2})$ to account for relatively small fluctuations in $\sigma_f^2(t)$. When written in the form of Eq. (1.11), the general form of the correction term $Q^{(2)}(y|\overline{\sigma_f^2})$ is given by Eq. (4.34) in the text.

For cases where the response of the aircraft to the slow component $w_s(t)$ is negligible in comparison with the response to the component $\sigma_f(t)\ z(t)$, the form of $N_+(y)$ given by Eq. (1.11) is particularly instructive. For these cases, when we

*Equations (1.6) and (1.10) provide motivation for expressing $N_+(y)$ as an integral over the probability density $p(\sigma_f^2)$ of the local *variance* σ_f^2 rather than as an integral over the probability density of the local *standard deviation* σ_f, as was done by Press and others. The first term in the series expansion of $N_+(y)$ given by Eq. (1.6) is, according to Eq. (1.10), $N_+(y|\overline{\sigma_f^2})$. However, $N_+(y|\overline{\sigma_f^2})$ is simply the exceedance rate obtained by assuming that $\sigma_f^2(t)$ is a constant. In cases where $\sigma_f(t)$ varies somewhat with time and we *estimate* $N_+(y)$ by *assuming* that the response is a stationary Gaussian process, we obtain for our estimate of $N_+(y)$ the first term $N_+(y|\overline{\sigma_f^2})$ in the right-hand sides of Eqs. (1.6) and (1.10) as is shown in Sec. 4.3 [see Eq. (4.32)]. However, since $E\{\sigma_f^2\} \neq \{E[\sigma_f]\}^2$, the quantity $N_+(y|\overline{\sigma_f^2})$ is different from the exceedance rate that would be obtained by evaluating the expression for the exceedance rate for stationary Gaussian processes using for σ_f the mean of the probability density of σ_f rather than the square root of $\overline{\sigma_f^2}$.

take the logarithm of Eq. (1.11) and assume that $\mu^{(2)}_{\sigma_f^2}$ is small enough so that we may use $\ell n(1+x) \approx x$, we obtain in Sec. 4.3 the simple relationship

$$\ell n \frac{N_+(y)}{N_+^{(0)}} \approx - \frac{y^2}{2\sigma_y^2} + \frac{1}{8} \frac{\mu^{(2)}_{\sigma_f^2}}{(\overline{\sigma_f^2})^2} \frac{y^2}{\sigma_y^2} \left(\frac{y^2}{\sigma_y^2} - 4\right). \qquad (1.13,\ 4.40)$$

The first term $(-y^2/2\sigma_y^2)$ on the right-hand side of Eq. (1.13) is the familiar result for stationary, Gaussian processes; i.e., the logarithm of the normalized exceedance rate $N_+(y)/N_+(0)$ is linear when plotted as a function of the square of the response level, and σ_y^2 is the variance of the response process. The coefficient $\mu^{(2)}_{\sigma_f^2}/(\overline{\sigma_f^2})^2$ governs the strength of the correction to the Gaussian approximation given by the first term. The quantity $\mu^{(2)}_{\sigma_f^2}/(\overline{\sigma_f^2})^2$ is the square of the coefficient of variation of the time-varying variance $\sigma_f^2(t)$ in the model of Eq. (1.1). Except for the value of $\mu^{(2)}_{\sigma_f^2}/(\overline{\sigma_f^2})^2$, which is always nonnegative, the functional form of the correction term in the right-hand side of Eq. (1.13) is fixed. For $y = 0$, the correction is zero; hence, the Gaussian approximation given by the first term yields the correct value. For $0 < y < (2\sigma_y)$, the correction term is negative; hence, in this interval, the Gaussian approximation *overestimates* the threshold-crossing rate. For $y > (2\sigma_y)$, the correction term is positive; hence, for large values of y the Gaussian approximation *underestimates* the threshold-crossing rate. These theoretical predictions are consistent with well known experimental observations, when considered as a function of y^2 (or y^2/σ_y^2), Eq. (1.13) predicts (a parabolic) concave exceedance plot.

A treatment of the probability density function p(y) of the aircraft response is provided in Sec. 4.4, where we have used the same general approach that was used for exceedance rates. For example, it is shown there that we may express p(y) by

$$p(y) = \int_0^\infty p(y|\sigma_f^2)p(\sigma_f^2)\,d\sigma_f^2 \ . \qquad (1.14,\ 4.42)$$

A series expansion for p(y) that is developed [Eq. (4.46)] is analogous to that of Eq. (1.6). The two-term approximation to this series expansion that is analogous to Eq. (1.1) is

$$p(y) \simeq p(y|\overline{\sigma_f^2})\,[1+\mu^{(2)}_{\sigma_f^2} U^{(2)}(y|\overline{\sigma_f^2})] \qquad (1.15,\ 4.51)$$

where $U^{(2)}(y|\overline{\sigma_f^2})$ is defined by

$$U^{(2)}(y|\overline{\sigma_f^2}) = \frac{1}{8}\left(\frac{\sigma^2_{y_z}}{\sigma_y^2}\right)^2\left[\left(\frac{y^2}{\sigma_y^2}\right)^2 - 6\,\frac{y^2}{\sigma_y^2} + 3\right], \qquad (1.16,\ 4.50)$$

where we have used the notation of Eq. (4.48). The quantity $\sigma^2_{y_z}$ is the aircraft mean-square response to turbulence component z(t) given by Eq. (4.18), and σ_y^2 is the overall mean-square response given by Eqs. (4.16) and (4.48). It is shown in in Sec. 4.4 that the correction term given in Eq. (1.15) to the Gaussian approximation $p(y|\overline{\sigma_f^2})$ of p(y) is exactly the same as the first correction term provided by the Gram-Charlier expansion. However, the derivation of Eq. (1.15) is based on what would appear to be a completely different line of reasoning.

For cases where the aircraft response to the "slow" turbulence component $w_s(t)$ is negligible, the correction term to the Gaussian approximation in Eq. (1.11) is given by

$$Q^{(2)}(y|\overline{\sigma_f^2}) = \frac{1}{8(\overline{\sigma_f^2})^2}\,\frac{y^2}{\sigma_y^2}\left(\frac{y^2}{\sigma_y^2} - 4\right). \qquad (1.17,\ 4.39)$$

Comparison of Eqs. (1.16) and (1.17) indicates that whenever the fluctuation in $\sigma_f^2(t)$ is not negligible in comparison with its mean $\overline{\sigma_f^2(t)}$, the first-order probability density p(y) and the exceedance rate $N_+(y)$ have different functional dependencies on y. However, when $\mu^{(2)}_{\sigma_f^2} = 0$, both p(y) and $N_+(y)$ have the shape of Gaussian probability density functions.

The locally stationary response requirements, initially treated in Sec. 3, are discussed and related to stochastic metrics of the process $\sigma_f(t)$ in Sec. 5. In particular, it is

shown in Sec. 5.1 that the locally stationary response requirements may be interpreted as requiring (1) that the relative changes in $\sigma_f^2(t)$ must occur slowly when measured on the time scale $t' = L_z/V$, where L_z is the integral scale of the component $z(t)$ in the model of Eq. (1.1) and V is the aircraft speed [see Eq. (5.1)]; (2) that the relative changes in $\sigma_f^2(t)$ must be small in comparison with unity when measured over time intervals of the order of the group delay of the aircraft impulse response function [see Eqs. (5.4) and (5.6)]; and (3) that the relative changes in $\sigma_f^2(t)$ must occur slowly relative to time intervals comparable to the nominal duration of $h^2(t)$, where $h(t)$ is the aircraft unit-impulse response function [see Eqs. (3.46) and (5.7)].

These three spectrum conditions are expressed in terms of (somewhat less stringent) mean-square response conditions by Eqs. (5.8), (5.9), and (5.10) in Sec. 5.2. In Sec. 5.3, the requirements on the behavior of $\sigma_f^2(t)$ are expressed in terms of derivatives of the autocorrelation function of the logarithm of $\sigma_f^2(t)$, where the dependence on the logarithm of $\sigma_f^2(t)$, rather than on $\sigma_f^2(t)$ itself, has been dictated by the requirements themselves. [It is the *fractional* fluctuation of $\sigma_f^2(t)$ that is important, rather than the *absolute* fluctuation of $\sigma_f^2(t)$]. The final forms of the three local stationarity requirements are*

$$[R_V^{(4)}(0)]^{\frac{1}{2}} \le 3.2\pi^2 \frac{\displaystyle\int_{-\infty}^{\infty} \Phi_z(f)|H(f)|^2 df}{\left|\displaystyle\int_{-\infty}^{\infty} \Phi_z^{(2)}(f)|H(f)|^2 df\right|} \qquad (1.18,\ 5.17)$$

$$[-R_V''(0)]^{\frac{1}{2}} \le \frac{1}{10} \frac{\displaystyle\int_{-\infty}^{\infty} \Phi_z(f) m_h^{(0)}(f) df}{\left|\displaystyle\int_{-\infty}^{\infty} \Phi_z(f) m_h^{(1)}(f) df\right|} \qquad (1.19,\ 5.18)$$

*One of the other forms of these requirements given in the main text in Sec. 3 or 5 may be easier to apply in practice. Equations (1.18), (1.19), and (1.20) would seem to be the least restrictive set of requirements.

and

$$-\frac{1}{3}R_v''(0)\left(1+3\left\{2+\frac{R_v^{(4)}(0)}{[R_v''(0)]^2}\right\}^{\frac{1}{2}}\right)\leq\frac{1}{5}\,\frac{\int_{-\infty}^{\infty}\Phi_z(f)m_h^{(0)}(f)df}{\left|\int_{-\infty}^{\infty}\Phi_z(f)m_h^{(2)}(f)df\right|}\,, \tag{1.20, 5.22}$$

where $R_v''(0)$ and $R_v^{(4)}(0)$ are the second and fourth deviatives of the autocorrelation function of

$$v(t) \triangleq \ell n\sigma_f^2(t), \tag{1.21, 5.11}$$

where these derivatives are to be evaluated at the origin, $\Phi_z(f)$ is the power spectrum of the component $z(t)$ in the model of Eq. (1.1), $\Phi_z^{(2)}(f)$ is the second derivative of $\Phi_z(f)$, $H(f)$ is the aircraft complex frequency-response function, and for $n = 0$, 1, and 2, $m_h^{(n)}(f)$ is the power-moment spectrum of the aircraft unit-impulse response function $h(t)$ defined in earlier work by this writer in a completely different context:

$$m_h^{(n)}(f) \triangleq \int_{-\infty}^{\infty} t^n\Phi_h(f,t)dt, \tag{1.22, 3.36}$$

where $\Phi_h(f,t)$ is the instantaneous spectrum of $h(t)$ defined by Eqs. (3.25) and (3.26). Writing the complex frequency-response function $H(f)$ as

$$H(f) = |H(f)|e^{i\theta_h(f)}, \tag{1.23, 5.3}$$

we express $m_h^{(0)}(f)$, $m_h^{(1)}(f)$, and $m_h^{(2)}(f)$ in terms of the magnitude and phase of $H(f)$ by

$$m_h^{(0)}(f) = |H(f)|^2 \tag{1.24, 3.39}$$

$$m_h^{(1)}(f) = -\frac{1}{2\pi}|H(f)|^2\frac{d\theta_h(f)}{df} \tag{1.25}$$

$$m_h^{(2)}(f) = \frac{1}{4\pi^2}\,|H(f)|^2 \left\{ \left[\frac{d\theta_h(f)}{df}\right]^2 - \frac{1}{2}\,\frac{d^2 \ell n|H(f)|}{df^2} \right\} , \quad (1.26)$$

where the time origin of $h(t)$ must be chosen to satisfy Eq. (3.22) and where Eqs. (1.25) and (1.26) are a consequence of Eqs. (3.39), (5.4), and (5.7). Equations (1.18), (1.19), and (1.20) would appear to be the least restrictive requirements that must be satisfied for confident engineering usage of the locally stationary response approximation of Eq. (1.4), which is the basic approximation used in deriving the above described expressions for the exceedance rates and probability density functions of an aircraft-response variable $y(t)$.

Evaluation of the left-hand side of the requirements of Eqs. (1.18), (1.19), and (1.20) requires a capability to evaluate from measured turbulence records the autocorrelation function $R_v(\tau)$ of $\ell n\sigma_f^2(t)$, as indicated by Eq. (1.21). Since only the derivatives of $R_v(\tau)$ are required, $R_v(\tau)$ need be determined only to within an additive constant. In Sec. 5.4, it is shown that $R_v(\tau)$ may be expressed as

$$R_v(\tau) = \{E[\ell n\ \sigma_f^2(t)]\}^2 + \mathrm{Cov}[\ell n\ w_h^2(t),\ \ell n\ w_h^2(t+\tau)]$$

$$- 2\ \arcsin^2\ [R_{z_h}(\tau)/R_{z_h}(0)], \qquad (1.27,\ 5.36)$$

where $\{E[\ell n\ \sigma_f^2(t)]\}^2$ is a constant, $\mathrm{Cov}[\cdots]$ is the covariance of $\ell n\ w_h^2(t)$ and $\ell n\ w_h^2(t+\tau)$, which can be evaluated directly from a high-pass filtered version $w_h(t)$ of the turbulence record $w(t)$ as described in Sec. 5.4, and $R_{z_h}(\tau)$ is the (inverse) Fourier transform of the high-pass filtered version of $\Phi_z(f)$ defined by Eq. (5.28). All of the above quantities are amenable to numerical calculation from turbulence velocity records. To obtain Eq. (1.27), we had to derive an expression for the autocorrelation function of the natural logarithm of the square of a stationary Gaussian process from the autocorrelation function of the process itself. This result is given (for processes with zero mean and unit variance) by Eq. (5.31).

The above work suggests that, in addition to the autocorrelation function $R_v(\tau)$ of $\ell n\ \sigma_f^2(t)$, an adequate characterization of turbulence records for aircraft response predictions requires the spectra $\Phi_{w_s}(f)$ and $\Phi_z(f)$ of the components $w_s(t)$ and $z(t)$ in the turbulence model of Eq. (1.1) and the probability

density function $p(\sigma_f^2)$ of the square of the component $\sigma_f(t)$ in Eq. (1.1). All of these quantities are required for computation of response exceedance rates - e.g., see Eqs. (1.4) and (1.5). In addition, it would be desirable to test the assumption that $\{w_s(t)\}$ is a (stationary) Gaussian process. The "first-order test" of this assumption is the probability density function of the component $w_s(t)$. Especially relevant metrics of the probability density functions of $\sigma_f^2(t)$ and $w_s(t)$ are the low-order central moments - e.g., see Eq. (1.13). Furthermore, it appears likely that the fourth-order moment of the response process $y(t)$ can be computed if the power spectrum of $\sigma_f^2(t)$ is available, even in cases where neither of the locally stationary response conditions of Eqs. (1.19) and (1.20) are satisfied, but where the requirement of Eq. (1.18) is satisfied.* Thus, in addition to the autocorrelation function of $\ln \sigma_f^2(t)$, the power spectra of $z(t)$, $w_s(t)$, and $\sigma_f^2(t)$ and the moments and probability density functions of $\sigma_f^2(t)$ and $w_s(t)$ are useful turbulence characterizations for the prediction of aircraft-response statistics. Methods for computation of these turbulence metrics are developed in Sec. 6. Specifically, methods for estimating the spectra of $w_s(t)$ and $z(t)$ are described in Sec. 6.1, methods for estimating the spectrum of $\sigma_f^2(t)$ are described in Sec. 6.2, and methods for estimating the moments and probability density functions of $\sigma_f^2(t)$ and $w_s(t)$ are described, respectively, in Secs. 6.3 and 6.4.

As is the case with most newly developed research results, the methods and conclusions reported herein represent a (hopefully error free) "first cut" at the problem of adequately taking into account - in a not too complicated fashion - the nonGaussian behavior of real turbulence records for the purpose of predicting aircraft-response statistics. Recommended future work would include "fine tuning" and improvements to methods and conclusions reported herein.

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*This result will be of use in the case of very high-speed aircraft.

NONGAUSSIAN TURBULENCE MODEL

Atmospheric turbulence velocity measurements are usually made in sets of three records: the vertical, lateral, and longitudinal time histories. When Taylor's hypothesis - i.e., $x = Vt$ - is employed, there result vertical, lateral, and longitudinal records that are regarded as functions of a spatial variable x. The "standard" turbulence model is to assume that each of these records is a sample function drawn from a stationary (or homogeneous) Gaussian random process. Furthermore, it is generally assumed that the power spectra of the vertical and lateral records should be well described by the von Karman transverse spectrum and that the longitudinal records should be well described by the von Karman longitudinal spectrum. Each of these two von Karman spectral forms is described by two parameters, the rms turbulence velocity σ and the integral scale L. Since the power spectral density provides a complete probabilistic description of a Gaussian random process, these von Karman spectral forms provide complete statistical descriptions of the three components of measured turbulence velocities - if the stationary Gaussian model with von Karman spectra is a valid model of turbulence velocity records.

However, results predicted by the above model are not always in agreement with turbulence measurements. Differences between the model and observed records are manifested in at least two ways: (1) the probabilities of large excursions are underestimated by the model, and (2) the low-frequency content of turbulence velocity spectra is often underestimated by the model. In addition, the "knee" of measured spectra is often less sharp than the knee of the appropriate von Karman spectral form.

The fact that the probability of large excursions is often underestimated by the standard turbulence model is directly attributable to the fact that turbulence time histories often appear to have a time-varying envelope or patchy character. One may regard such time histories to be either nonstationary but Gaussian or stationary but nonGaussian. For a given standard deviation (rms value) of turbulence velocity, a time variation in the envelope generally has the effect of yielding more large excursions than the Gaussian probability density predicts. This behavior accounts for the "concave shape" of turbulence exceedance plots.

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*The distinction between nonstationary Gaussian behavior and stationary nonGaussian behavior is, in general, impossible to make with a single time history. It is often more *convenient* to regard such records to be stationary but nonGaussian (or locally Gaussian).

The above comments suggest that a given turbulence record, say $w(t)$, may be modeled by

$$w(t) = \sigma(t)\, z(t), \qquad \sigma(t) \geq 0 \tag{2.1}$$

$$E\{z\} = 0, \qquad E\{z^2\} = 1, \tag{2.2}$$

where $\sigma(t)$ may be regarded as a time-varying standard deviation (or envelope), and where $\{z(t)\}$ is a stationary process, with zero mean value and unit standard deviation, which may be taken to be Gaussian. Time variations of the function $\sigma(t)$ account for the patch-like character of atmospheric turbulence velocities observed in many records. Insofar as the visual appearance of the velocity records is concerned, the function $\sigma(t)$ may be regarded as either a stochastic or a deterministic function. Appropriate choice of the amplitude distribution of $\sigma(t)$ will allow the probability distribution of $w(t)$ to take on a wide variety of forms.

Visual inspection of turbulence records indicates one additional feature that cannot be accounted for by the model of Eq. (2.1). Some records which exhibit the patch-like character modeled by Eq. (2.1) also have *superimposed* on this patch-like structure another very low frequency component, which appears to fluctuate independently of the envelope of the higher-frequency components that possess the patch-like character. This apparent independence suggests that a low-frequency term be added to Eq. (2.1); i.e.,

$$\begin{aligned} w(t) &= w_s(t) + w_f(t) \\ &= w_s(t) + \sigma_f(t)\, z(t), \end{aligned} \tag{2.3}$$

where

$$w_f(t) = \sigma_f(t)\, z(t), \qquad \sigma_f(t) \geq 0 \tag{2.4}$$

and

$$E\{z\} = 0, \quad E\{z^2\} = 1. \tag{2.5}$$

In the following work, we shall assume that $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$ is each a stationary random process and that $\{z(t)\}$ is a Gaussian process with zero mean value and unit variance, as indicated by Eq. (2.5). The spectrum of the "slow" process $\{w_s(t)\}$ generally occupies a frequency range that is low in comparison with that occupied by the "fast"

process $\{w_f(t)\}$; it is the low-frequency contribution of $\{w_s(t)\}$ that causes underprediction by the von Karman model of the low-frequency part of the spectrum. Thus, we shall further assume that the power spectrum of $\{z(t)\}$ is described by the appropriate von Karman form and that $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$ are mutually independent processes. In some of our work, we shall also assume that $\{w_s(t)\}$ is a zero mean Gaussian process. However, we shall *not* want to assume that $\{\sigma(t)\}$ is Gaussian, since such an assumption would permit $\sigma(t)$ to go negative. *Equation (2.3) is the simplest model of atmospheric turbulence that possesses the flexibility required to represent readily observable features of measured turbulence records.*

At this juncture, it is appropriate to illustrate the need for the model of Eq. (2.3) by examination of some measured turbulence velocity histories. The vertical record shown in Fig. 1 (Ref. 1) illustrates a record that is probably reasonably well modeled by a stationary Gaussian process, especially the right-hand half of the record - i.e., from 120 sec elapsed time to the end. Thus, this record would be reasonably modeled by Eq. (2.3) with $\sigma_f(t)$ set equal to a constant - in which case there is no need for the term $w_s(t)$.

The records shown in Fig. 2 (Ref. 1) illustrate mild patch-like behavior. For example, the patches occurring at 150 and 160 sec elapsed time on the vertical record illustrate distinctly nonstationary or nonGaussian behavior. The number of large excursions of the records shown in Fig. 2 is substantially larger than would occur for stationary Gaussian records with the same standard deviations and spectra.

Each of the records shown in Fig. 2 also exhibits an additive low-frequency component that appears to fluctuate independently of the occurrence of the patches. For example, for the 5-sec interval between 183 and 188 sec on the vertical record, high-frequency fluctuations are absent; however, there remains in that interval a fluctuating low-frequency component. Similar behavior occurs between approximately 96 and 99 sec elapsed time on the vertical record shown in Fig. 3 (Ref. 1). A strong low-frequency component is present there; however, high-frequency fluctuations are absent in that interval. These "gaps" cannot be explained in terms of statistical fluctuations of a stationary Gaussian process.

Each of the records shown in Fig. 4 (Ref. 2) dramatically illustrates the three components of the turbulence model of Eq. (2.3). For example, in the region from 9 min 0 sec to 9 min 45 sec on the vertical record, the term $\sigma_f(t)$ in Eq. (2.3) is negligible in comparison with the very strong low-frequency component $w_s(t)$, which is clearly present over the entire

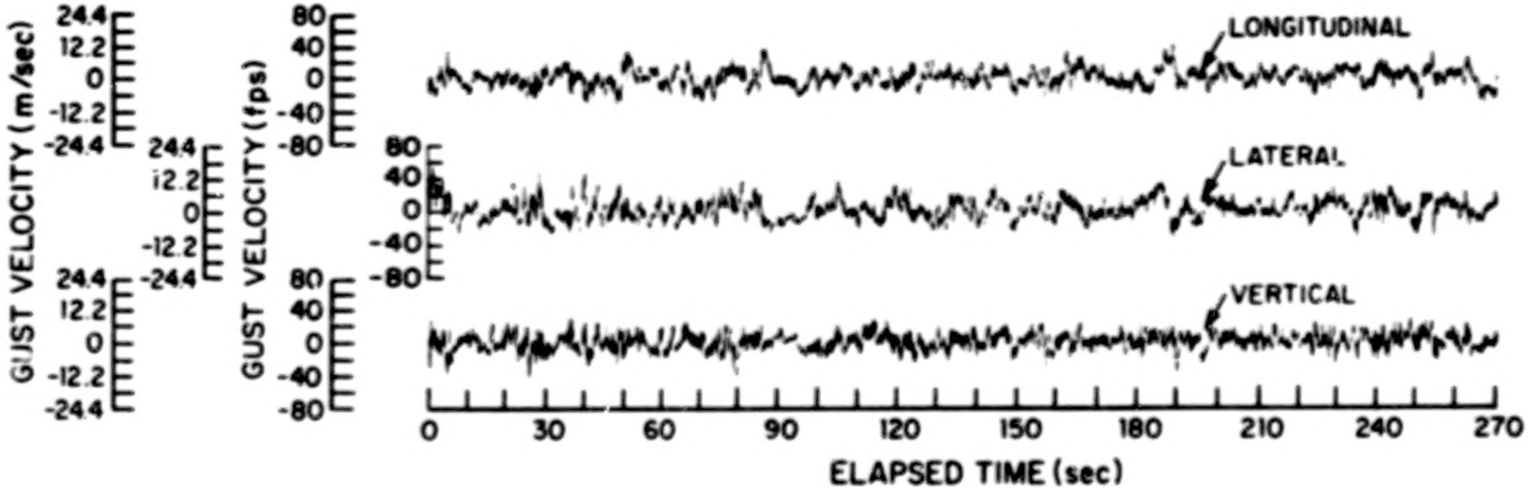

FIG. 1. THE VERTICAL RECORD FROM 120 TO 270 SEC ELAPSED TIME ILLUSTRATES REASONABLY STATIONARY GAUSSIAN BEHAVIOR. (Ref. 1, Fig. 17.32, p. 223.)

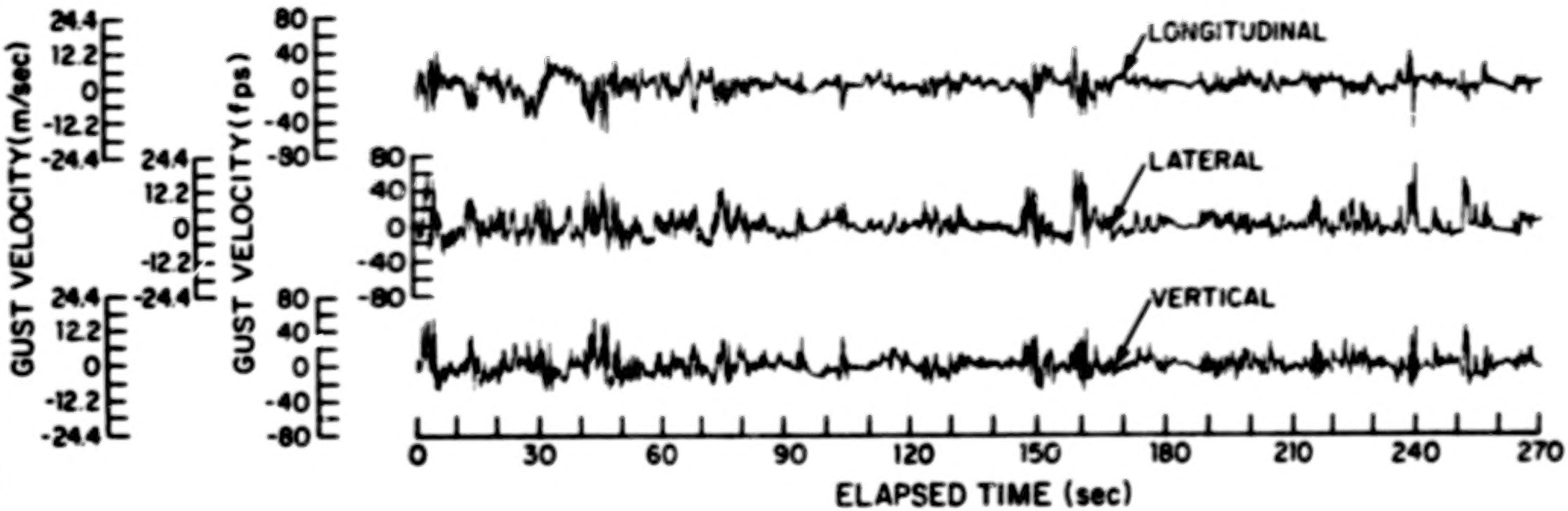

FIG. 2. THE ABOVE RECORDS ILLUSTRATE MILD PATCH-LIKE NONGAUSSIAN BEHAVIOR OF ATMOSPHERIC TURBULENCE. (Ref. 1, Fig. 17.28, p. 219.)

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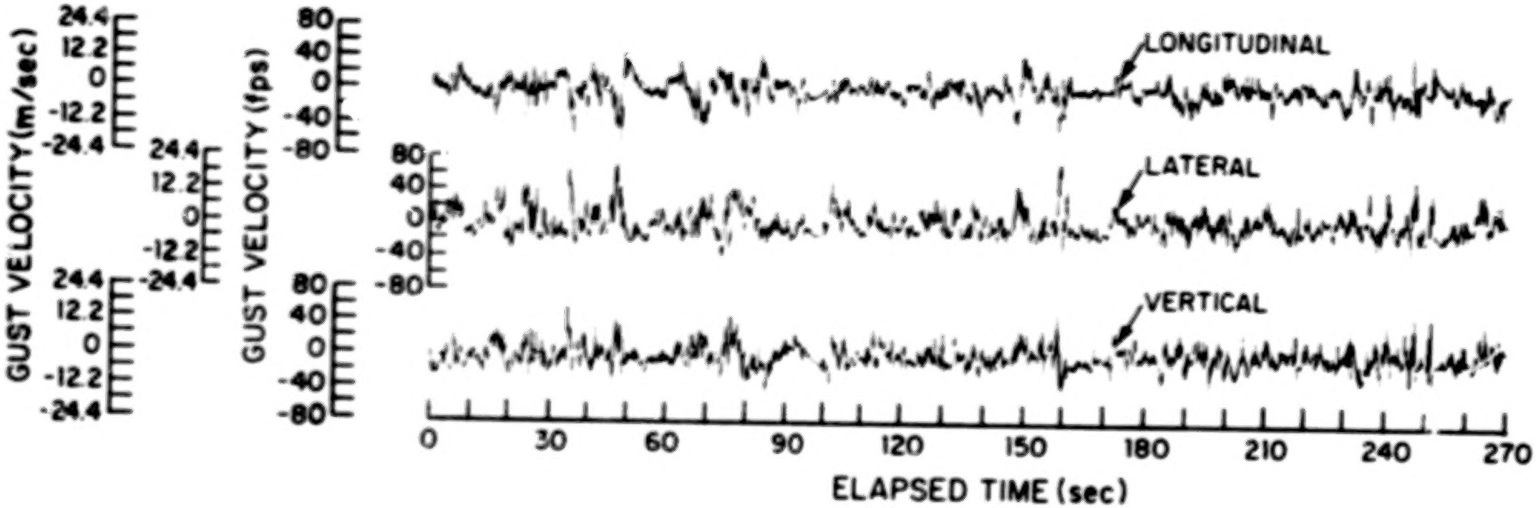

FIG. 3. AN ADDITIVE LOW-FREQUENCY COMPONENT IS EVIDENT IN THE VERTICAL RECORD. (Ref. 1, Fig. 17.34, p. 225.)

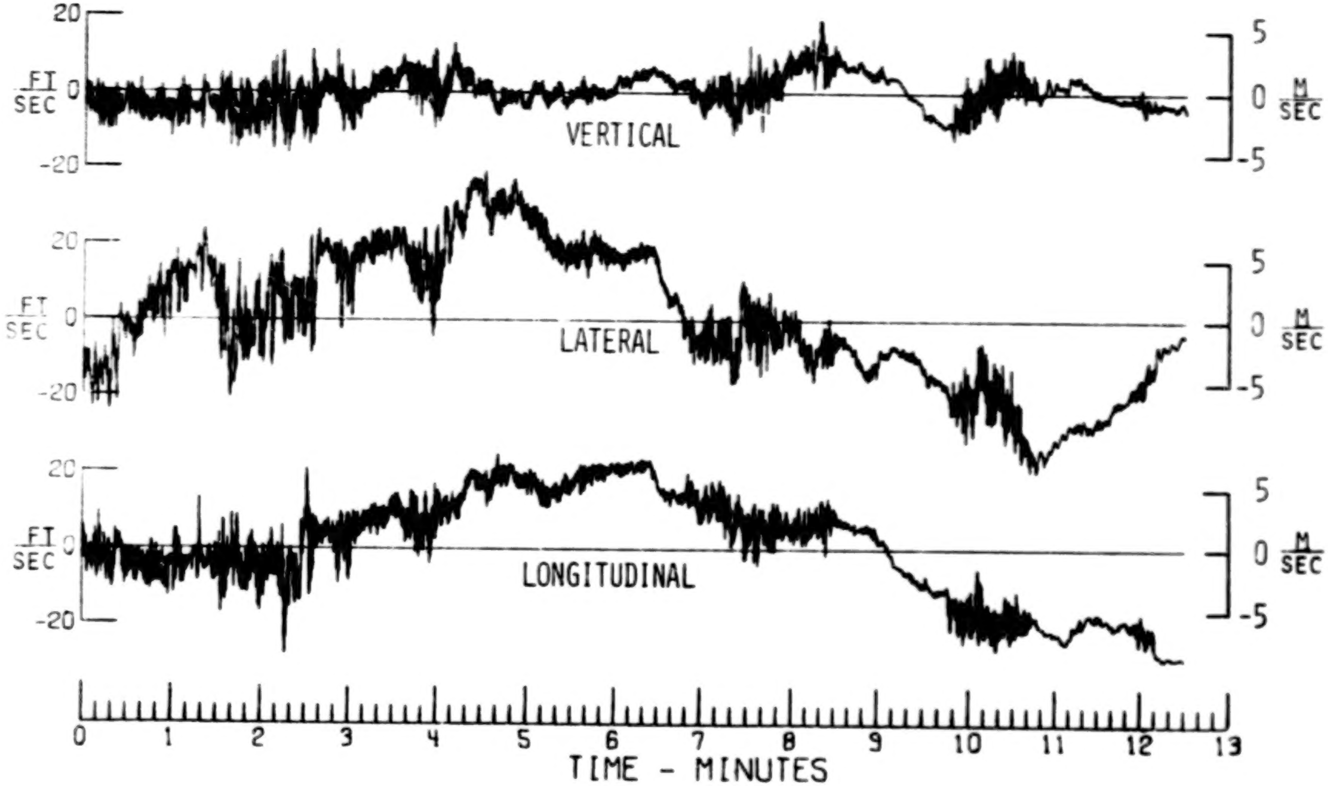

FIG. 4. EACH OF THE ABOVE RECORDS CLEARLY ILLUSTRATES THE INDIVIDUAL COMPONENTS $w_s(t)$, $\sigma_f(t)$, and $z(t)$ IN THE THREE COMPONENT MODEL OF EQ. (2.3). [MOUNTAIN WAVE CONDITIONS. AIRCRAFT SPEED 197 m/sec (646 ft/sec).] (Ref. 2, Fig. 10, p. 285.)

record. At about 9 min 45 sec, $\sigma_f(t)$ grows and then decays back to a small value about one minute later. The behavior of the records shown in Fig. 4 *cannot* be modeled by a single stationary Gaussian process.

A model functionally similar to Eq. (2.3) has been suggested by Reeves *et al.* [3,4] for use with flight simulators. However, in Reeves' model, $\{\sigma(t)\}$ and $\{z(t)\}$ are specified as (stationary) Gaussian processes, and the processes $\{w_s(t)\}$ and $\{w_f(t)\}$ both have the same spectral form – the Dryden spectrum – and both have the same integral scale. The model of Eq. (2.1) has been proposed independently by Sidwell [5] and Mark [6]. In addition, Sidwell in work currently being carried out at NASA Langley Research Center, has proposed the addition of the low-frequency component $w_s(t)$ indicated by the model of Eq. (2.3).

The model of Eq. (2.3) could be generalized further – e.g., by including a multiplicative modulating term $\sigma_s(t)$ in the slow component $w_s(t)$. In this case, the slow and fast components $w_s(t)$ and $w_f(t)$ would have identical functional forms. This generalization, which may have to be made after further examination of additional turbulence records, requires only modest changes in the methods developed in this report.

Conditional Instantaneous Spectrum of Turbulence Model

In the turbulence velocity model described by Eq. (2.3), the three stochastic functions $w_s(t)$, $\sigma_f(t)$, and $z(t)$ are assumed to be mutually independent sample functions drawn from the three processes $\{w_s(t)\}$, $\{\sigma_f(t)\}$, and $\{z(t)\}$. For reasons that will become evident later, we shall be concerned here with the conditional instantaneous spectrum of the stochastic function defined by Eq. (2.3); i.e.,

$$w(t) = w_s(t) + \sigma_f(t)\, z(t) \quad , \tag{3.1}$$

where this instantaneous spectrum will be conditioned on the *stochastic function* $\sigma_f(t)$. The operation of forming the conditional expectation required in developing the instantaneous spectrum of $w(t)$ is equivalent to treating $\sigma_f(t)$ as a known (or deterministic) function of completely general form, except for the requirement that $\sigma_f(t) \geq 0$.*

To determine an expression for the conditional instantaneous spectrum of $w(t)$, we first form the conditional instantaneous autocorrelation function of $w(t)$. The required instantaneous autocorrelation function $\phi_w(\tau,t|\sigma_f)$ is defined – e.g., Mark [7], Mark and Fischer [6] – as

$$\phi_w(\tau,t|\sigma_f) \triangleq E\{w(t-\tfrac{\tau}{2})\; w(t+\tfrac{\tau}{2})|\sigma_f(t)\} \tag{3.2a}$$

$$= E\{[w_s(t-\tfrac{\tau}{2}) + \sigma_f(t-\tfrac{\tau}{2})\; z(t-\tfrac{\tau}{2})]$$

$$\times\; [w_s(t+\tfrac{\tau}{2}) + \sigma_f(t+\tfrac{\tau}{2})\; z(t+\tfrac{\tau}{2})]|\sigma_f(t)\} \quad , \tag{3.2b}$$

where the vertical bars followed by $\sigma_f(t)$ in the left- and right-hand sides of Eq. (3.2) indicate that the expectation operation $E\{\ldots|\sigma_f(t)\}$ assumes that the (stochastic) function $\sigma_f(t)$ is known or specified; hence, no averaging is carried out over the ensemble of functions $\{\sigma_f(t)\}$. See, for example, pp. 55 and 56 of Laning and Battin [8] for a brief discussion of the notion of a conditional expectation. (Chapters 1 through 5 of this reference provide an excellent introductory discussion of probability theory and stochastic processes.) Expanding the right-hand side of Eq. (3.2b)

*The concept of the process $w_f(t)$ conditioned on the process $\sigma_f(t)$ has been used by Sidwell [5].

and recognizing that $\{w_s(t)\}$ and $\{z(t)\}$ are independent stationary processes, that $E\{z\} = 0$, and that $\sigma_f(t)$ is being treated as a known function, we have

$$\phi_w(\tau,t|\sigma_f) = E\{w_s(t-\tfrac{\tau}{2})\, w_s(t+\tfrac{\tau}{2})\}$$

$$+ \sigma_f(t-\tfrac{\tau}{2})\, \sigma_f(t+\tfrac{\tau}{2})\, E\{z(t-\tfrac{\tau}{2})\, z(t+\tfrac{\tau}{2})\}$$

$$= \phi_{w_s}(\tau) + \phi_{\sigma_f}(\tau,t|\sigma_f)\, \phi_z(\tau) \quad , \tag{3.3}$$

where $\phi_{w_s}(\tau)$ and $\phi_z(\tau)$ are the autocorrelation functions of $\{w_s(t)\}$ and $\{z(t)\}$ and where we have defined

$$\phi_{\sigma_f}(\tau,t|\sigma_f) \triangleq \sigma_f(t-\tfrac{\tau}{2})\, \sigma_f(t+\tfrac{\tau}{2}) \quad , \tag{3.4}$$

which requires no stochastic average since $\sigma_f(t)$ is assumed to be specified.

The (conditional) instantaneous spectrum of w(t) is defined [e.g., 6,7] as the Fourier transform with respect to τ of $\phi_w(\tau,t|\sigma_f)$:

$$\Phi_w(f,t|\sigma_f) \triangleq \int_{-\infty}^{\infty} \phi_w(\tau,t|\sigma_f)\, e^{-i2\pi f\tau}\, d\tau \quad . \tag{3.5}$$

Recognizing that the Fourier transform of a product is the convolution of the transforms, and treating t as a parameter in Eq. (3.3), the conditional instantaneous spectrum of w(t) may be immediately obtained from Eq. (3.3) as

$$\Phi_w(f,t|\sigma_f) = \Phi_{w_s}(f) + \int_{-\infty}^{\infty} \Phi_{\sigma_f}(g,t|\sigma_f)\, \Phi_z(f-g)\, dg \quad , \tag{3.6}$$

where $\Phi_{w_s}(f)$ and $\Phi_z(f)$ are the power spectra of the stationary processes $\{w_s(t)\}$ and $\{z(t)\}$, which are obtained by forming the Fourier transforms of the appropriate autocorrelation functions — i.e.,

$$\Phi_{\cdot}(f) \triangleq \int_{-\infty}^{\infty} \phi_{\cdot}(\tau)\, e^{-i2\pi f\tau}\, d\tau \tag{3.7}$$

– and where the conditional instantaneous spectrum $\Phi_{\sigma_f}(f,t|\sigma_f)$ in Eq. (3.6) is obtained by forming the Fourier transform with respect to τ of $\phi_{\sigma_f}(\tau,t|\sigma_f)$. Equation (3.6) is the desired expression for the conditional instantaneous spectrum of the process $\{w(t)\}$ defined by Eq. (3.1).

Series Expansion of Conditional Instantaneous Spectrum of Turbulence Model

Equation (3.6) can be written as

$$\Phi_w(f,t|\sigma_f) = \Phi_{w_s}(f) + \Phi_{w_f}(f,t|\sigma_f) \quad , \tag{3.8}$$

where

$$\Phi_{w_f}(f,t|\sigma_f) = \int_{-\infty}^{\infty} \Phi_{\sigma_f}(g,t|\sigma_f)\, \Phi_z(f-g)\, dg \tag{3.9}$$

is the conditional instantaneous spectrum of the component $w_f(t)$ defined by Eq. (2.4); i.e.,

$$w_f(t) = \sigma_f(t)\, z(t) \quad . \tag{3.10}$$

If the temporal variations in $\sigma_f(t)$ occur slowly in comparison with those of $z(t)$, we would expect the conditional instartaneous spectrum of $w_f(t)$ to have the form

$$\Phi_{w_f}(f,t|\sigma_f) \simeq \sigma_f^2(t)\, \Phi_z(f) \quad , \tag{3.11}$$

where $\Phi_z(f)$ is the power spectral density of the process $\{z(t)\}$. *Equation (3.11) is the locally stationary approximation for the conditional instantaneous spectrum of the "fast" component of turbulence $w_f(t)$.*

For most measured turbulence records, the fluctuations of $\sigma_f(t)$ do, in fact, vary slowly in comparison with those of $z(t)$.

Mark and Fischer [6] derived a series representation of the conditional instantaneous spectrum $\Phi_{w_f}(f,t|\sigma_f)$ that enables one to determine the conditions required for the approximation of Eq. (3.11) to be valid. This series representation was developed in terms of distance x and wavenumber k, rather than in terms of time t and frequency f. However, the results of Ref. 6 may be applied directly to the variables t and f simply by substituting t for x and f for k. Applying this substitution, we find from Eq. (4.11) of Ref. 6, with minor changes in notation,

$$\Phi_{w_f}(f,t|\sigma_f) = \sum_{n=0}^{N} \frac{a_n(t)}{n!} \Phi_z^{(n)}(f) + R_{N+1}(f,t) \quad , \tag{3.12}$$

where, from Eq. (4.9) of Ref. 6, $\Phi_z^{(n)}(f)$ is defined as the *nth* derivative of the power spectral density $\Phi_z(f)$ of $\{z(t)\}$ – i.e.,

$$\Phi_z^{(n)}(f) \triangleq \frac{d^n}{df^n} \Phi_z(f) \tag{3.13}$$

– and where

$$\Phi_z^{(0)}(f) = \Phi_z(f) \quad . \tag{3.14}$$

The coefficients $a_n(t)$ in Eq. (3.12) may be expressed in terms of the derivatives of $\sigma_f(t)$ by

$$a_n(t) = \frac{1}{(-i4\pi)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{d^k\sigma_f(t)}{dt^k} \frac{d^{n-k}\sigma_f(t)}{dt^{n-k}} \quad , \tag{3.15}$$

according to Eq. (4.19) of Ref. 6. The quantities $\binom{n}{k}$ are the binomial coefficients. From the above expression for $a_n(t)$, one may show that for odd integer values of n,

$$a_n(t) = 0 \quad , \quad n = \text{odd} \quad . \tag{3.16}$$

Expressions for the remainder term $R_{N+1}(f,t)$ in Eq. (3.12) are given by Eqs. (4.7) and (4.13) of Ref. 6. We are particularly interested in the first two nonvanishing terms $a_n(t)$ which, according to Eqs. (4.50) and (4.52) of Ref. 6, may be expressed as

$$a_0(t) = \sigma_f^2(t) \tag{3.17}$$

$$a_2(t) = -\frac{1}{8\pi^2}\,\sigma_f^2(t)\,\frac{d^2\ln\sigma_f(t)}{dt^2} \tag{3.18a}$$

$$= -\frac{1}{16\pi^2}\,\sigma_f^2(t)\,\frac{d^2\ln[\sigma_f^2(t)]}{dt^2} \quad . \tag{3.18b}$$

Combining Eqs. (3.12), (3.14), (3.16), (3.17), and (3.18), we have for the first two nonvanishing terms in our series expansion for $\Phi_{w_f}(f,t|\sigma_f)$

$$\Phi_{w_f}(f,t|\sigma_f) = \sigma_f^2(t)\left[\Phi_z(f) - \frac{1}{32\pi^2}\,\frac{d^2\ln\sigma_f^2(t)}{dt^2}\,\Phi_z^{(2)}(f)\right] + \dots \quad . \tag{3.19}$$

The first term in Eq. (3.19) is the locally stationary spectrum approximation given by Eq. (3.11). Whenever the second term in Eq. (3.19) is negligible in comparison with the first, the locally stationary approximation is valid. This condition may be expressed as

$$\left|\frac{d^2\ln\sigma_f^2(t)}{dt^2}\right| << 32\pi^2\,\frac{\Phi_z(f)}{|\Phi_z^{(2)}(f)|} \quad . \tag{3.20}$$

The above condition will be expressed in terms of measurable turbulence velocity metrics in a later section of this report.

By combining Eqs. (3.8) and (3.19), we obtain the series approximation to the conditional instantaneous spectrum of both turbulence components; i.e.,

$$\Phi_w(f,t|\sigma_f) = \Phi_{w_s}(f) + \sigma_f^2(t)\left[\Phi_z(f) - \frac{1}{32\pi^2}\,\frac{d^2\ln\sigma_f^2(t)}{dt^2}\,\Phi_z^{(2)}(f)\right] + \dots \quad , \tag{3.21}$$

which is the desired result. Notice that the correction term to the locally stationary approximation involves the second time derivative of $\ln\sigma_f^2(t)$. Thus, when $\ln\sigma_f^2(t)$ fluctuates sufficiently slowly – i.e., when Eq. (3.20) is satisfied – the locally stationary approximation is valid.

Conditional Instantaneous Spectrum of Aircraft Response

Consider an aircraft with spatial dimensions negligible in comparison with the scale of the von Karman component $z(t)$ of the turbulence model of Eq. (2.3). We may characterize a desired aircraft response variable by the response $h(t)$ of this variable to a "unit impulse" of turbulence velocity $w(t)$ applied at $t = 0$. However, for reasons that will become evident later, we shall want to choose the position of the time origin of $h(t)$ so that the time centroid of $h^2(t)$ occurs at $t = 0$; i.e., the time origin shall be chosen so that $h(t)$ satisfies

$$\frac{\int_{-\infty}^{\infty} t\ h^2(t)dt}{\int_{-\infty}^{\infty} h^2(t)dt} = 0 \quad . \tag{3.22}$$

Equation (3.22) defines a unique position for the time origin of $h(t)$; it is located at the center-of-gravity of the "mass distribution" $h^2(t)$.

Let us denote the conventionally defined unit-impulse response function by $\bar{h}(t)$, where the time origin of $\bar{h}(t)$ is generally chosen so that $\bar{h}(t) = 0$ for $t < 0$. The time centroid of $\bar{h}^2(t)$, which we shall denote by $\bar{t}_{\bar{h}}$, is defined by

$$\bar{t}_{\bar{h}} \triangleq \frac{\int_{-\infty}^{\infty} t\ \bar{h}^2(t)dt}{\int_{-\infty}^{\infty} \bar{h}^2(t)dt} \tag{3.23}$$

Our definition of $h(t)$ that satisfies Eq. (3.22) may be related to the conventionally defined $\bar{h}(t)$ by

$$h(t) = \bar{h}(t+\bar{t}_{\bar{h}}) \quad , \tag{3.24}$$

which can easily be verified by substitution of Eq. (3.24) into Eq. (3.22), introducing a change of variable, and then solving for $\bar{t}_{\bar{h}}$. The result of these operations yields Eq. (3.23).

To derive the desired input-response relationships for aircraft, we define the instantaneous autocorrelation function of the (deterministic) unit-impulse response function h(t) as

$$\phi_h(\tau,t) \triangleq h(t-\tfrac{\tau}{2})\, h(t+\tfrac{\tau}{2}) \quad , \tag{3.25}$$

from which we may define the instantaneous spectrum of h(t) as

$$\Phi_h(f,t) \triangleq \int_{-\infty}^{\infty} \phi_h(\tau,t)e^{-i2\pi f\tau}\, d\tau \quad . \tag{3.26}$$

Let {y(t)} denote the generally nonstationary response process of the aircraft, and let $\phi_y(\tau,t|\sigma_f)$ and $\Phi_y(f,t|\sigma_f)$ denote the conditional instantaneous autocorrelation function and conditional instantaneous spectrum of the response. Then, in Mark [7] it is shown that the conditional instantaneous autocorrelation function and spectrum of the response process are related to the corresponding characterizations of the input process w(t) and aircraft by

$$\phi_y(\tau,t|\sigma_f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_h(\xi,u)\ \phi_w(\tau-\xi,t-u|\sigma_f)d\xi du \tag{3.27}$$

and

$$\Phi_y(f,t|\sigma_f) = \int_{-\infty}^{\infty} \Phi_h(f,u)\ \Phi_w(f,t-u|\sigma_f)du \quad . \tag{3.28}$$

Substitution of Eq. (3.8) into Eq. (3.28) yields an expression for the conditional instantaneous response spectrum in terms of the spectra of the "slow" and "fast" components – i.e., $w_s(t)$ and $w_f(t)$ – of our turbulence model:

$$\begin{aligned}\Phi_y(f,t|\sigma_f) &= \int_{-\infty}^{\infty} \Phi_h(f,u)\ \Phi_{w_s}(f)du + \int_{-\infty}^{\infty}\Phi_h(f,u)\ \Phi_{w_f}(f,t-u|\sigma_f)du \\ &= \Phi_{w_s}(f)\ |H(f)|^2 + \int_{-\infty}^{\infty}\Phi_h(f,u)\ \Phi_{w_f}(f,t-u|\sigma_f)du \ ,\end{aligned} \tag{3.29}$$

where, in going to the second line, we have used Eq. 14 on p. 26 of Ref. 7 - applied to the deterministic function $h(t)$ - i.e.,

$$\int_{-\infty}^{\infty} \Phi_h(f,t)dt = |H(f)|^2 \quad , \tag{3.30}$$

where

$$H(f) \triangleq \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\, dt \tag{3.31}$$

is the aircraft frequency-response function.

Equation (3.29) is the desired spectrum input-response relationship. Notice that the first term in the right-hand side is the contribution to the response spectrum from the "slow" turbulence component $w_s(t)$ and that this contribution has the usual form of a response spectrum.

Series Expansion of Conditional Instantaneous Spectrum of Aircraft Response

By substituting Eq. (3.12) into Eq. (3.29), and then performing the indicated integration term by term, we obtain a series expansion for the conditional instantaneous response spectrum $\Phi_y(f,t|\sigma_f)$:

$$\begin{aligned}\Phi_y(f,t|\sigma_f) &= \Phi_{w_s}(f)\ |H(f)|^2 \\ &+ \sum_{n=0}^{N} \frac{\Phi_z^{(n)}(f)}{n!} \int_{-\infty}^{\infty} \Phi_h(f,u)\ a_n(t-u)du \\ &+ \int_{-\infty}^{\infty} \Phi_h(f,u)\ R_{N+1}\ (f,t-u)du \quad ,\end{aligned} \tag{3.32}$$

where the third line in the right-hand side of the above expression is the contribution of the remainder term.

The requirement of Eq. (3.20) is that the fluctuations of $\sigma_f^2(t)$ occur slowly in comparison with the fluctuations of the von Karman component z(t). This locally stationary requirement involves properties only of the turbulence. Obtaining our locally stationary representation of the aircraft response spectrum $\Phi_y(f,t|\sigma_f)$ further requires that fluctuations in $\sigma_f^2(t)$ be typically negligible over time intervals comparable to the nominal duration of the aircraft impulse response function h(t). We shall now derive explicit conditions that must be satisfied for a locally stationary response approximation to be valid.

First, we recall from Eqs. (3.15) to (3.18) that the terms $a_n(t)$ in Eq. (3.32) involve only the time behavior of the time-varying standard deviation $\sigma_f(t)$ of the "fast" turbulence component $w_f(t)$ – see, e.g., Eq. (2.4). If $\sigma_f(t)$ varies slowly in comparison with the nominal duration of h(t), then a few terms in the Taylor's series expansion of $a_n(t-u)$ about the instant u = 0 should provide a good approximation to $a_n(t-u)$ in Eq. (3.32) over the interval of u in the integral $\int_{-\infty}^{\infty} \Phi_n(f,u)\, a_n(t-u)du$, where $|\Phi_h(f,u)|$ is not negligible. This Taylor's series expansion of $a_n(t-u)$ is

$$a_n(t-u) = a_n(t) - u\, a_n'(t) + \frac{u^2}{2} a_n''(t) - \cdots \tag{3.33a}$$

$$= \sum_{k=0}^{M} \frac{(-1)^k}{k!} a_n^{(k)}(t) u^k + R_{M+1}(t,u) \quad , \tag{3.33b}$$

where

$$a_n^{(k)}(t) = \frac{d^k a_n(t)}{dt^k} \quad , \tag{3.34a}$$

$$a_n^{(0)}(t) = a_n(t) \quad , \tag{3.34b}$$

and where the magnitude of the remainder term satisfies

$$|R_{M+1}(t,u)| \le \max_{\xi} |a_n^{(M+1)}(\xi)| \frac{|u|^{M+1}}{(M+1)!} \quad . \tag{3.35}$$

Here, $\max_{\xi}|a_n^{(M+1)}(\xi)|$ designates the maximum value of the magnitude of $a_n^{(M+1)}(\xi)$ when ξ is varied over the range of values $(t-u) \le \xi \le t$

or $t \leq \xi \leq (t-u)$, depending on whether u is positive or negative, respectively. We may now substitute Eq. (3.33b) into Eq. (3.32) and integrate term by term. Defining

$$m_h^{(k)}(f) \triangleq \int_{-\infty}^{\infty} u^k \Phi_h(f,u)du \quad , \tag{3.36}$$

we find, upon carrying out this substitution and the resulting integration,

$$\begin{aligned}\Phi_y(f,t|\sigma_f) = {} & \Phi_{w_s}(f)\,|H(f)|^2 \\ & + \sum_{n=0}^{N} \frac{\Phi_z^{(n)}(f)}{n!} \sum_{k=0}^{M} \frac{(-1)^k}{k!}\, a_n^{(k)}(t)\, m_h^{(k)}(f) \\ & + \sum_{n=0}^{N} \frac{\Phi_z^{(n)}(f)}{n!} \int_{-\infty}^{\infty} \Phi_h(f,u)\, R_{M+1}(t,u)du \\ & + \int_{-\infty}^{\infty} \Phi_h(f,u)\, R_{N+1}(f,t-u)du \quad ,\end{aligned} \tag{3.37}$$

where the last two lines are remainder terms that are of academic interest only insofar as the present study is concerned. Equation (3.37) is the desired series representation of the conditional instantaneous aircraft response spectrum.

Requirements for Validity of Locally Stationary Aircraft Response Approximation

The purpose of the present study is to determine conditions under which all terms in Eq. (3.37) are negligible except for those corresponding to the locally stationary response approximation. To accomplish this goal, we write out the double summation in the second line in the right-hand side of Eq. (3.37) and make use of Eqs. (3.16) and (3.17). These operations yield

$$\Phi_y(f,t|\sigma_f) = \Phi_{w_s}(f)\ |H(f)|^2$$

$$+ \Phi_z(f)\left[\sigma_f^2(t)\ |H(f)|^2 - \frac{d}{dt}\sigma_f^2(t)\ m_h^{(1)}(f) + \frac{1}{2}\frac{d^2}{dt^2}\sigma_f^2(t) m_h^{(2)}(f) - \cdots\right]$$

$$+ \frac{1}{2}\Phi_z^{(2)}(f)\left[a_2(t)\ |H(f)|^2 - \frac{d}{dt}a_2(t) m_h^{(1)}(f) + \frac{1}{2}\frac{d^2}{dt^2}a_2(t) m_h^{(2)}(f) - \cdots\right]$$

$$+ \cdots , \tag{3.38}$$

where the first line on the right-hand side is the contribution to the response spectrum $\Phi_y(f,t|\sigma_f)$ from the "slow" turbulence component $w_s(t)$, the second line is the expansion over $k = 0,1,2,\ldots$ of the term $n = 0$ in the double summation in Eq. (3.37), the third line is the expansion over $k = 0,1,2,\ldots$ of the term $n = 2$ in the double summation in Eq. (3.37), and the $+\ldots$ in the last line represents terms $n > 2$ in the summation over n together with the remainder terms in the last two lines of Eq. (3.37). Also, in writing out Eq. (3.38), we have used the relationship

$$m_h^{(0)}(f) = |H(f)|^2 , \tag{3.39}$$

which is a direct consequence of Eqs. (3.30) and (3.36). We shall discuss interpretation of the quantities $m_h^{(k)}(f)$ later in this report.

The first line in the right-hand side of Eq. (3.38) requires no further discussion. The second line, which contains terms corresponding to $n = 0$ from Eq. (3.37), is the contribution to $\Phi_y(f,t|\sigma_f)$ from the locally stationary approximation to the conditional turbulence spectrum $\Phi_{w_f}(f,t|\sigma_f)$. This interpretation may be seen from Eqs. (3.12) and (3.19), where we remind the reader that the index n in Eqs. (3.12) and (3.37) plays the same role. The third line in the right-hand side of Eq. (3.38), which contains terms corresponding to $n = 2$ from Eq. (3.37), contains the contributions to $\Phi_y(f,t|\sigma_f)$ from the first correction term to the locally stationary approximation to the conditional turbulence spectrum $\Phi_{w_f}(f,t|\sigma_f)$. Furthermore, the first term within each of the brackets in the second and third lines in Eq. (3.38) provides the locally stationary approximation to the conditional response spectrum, whereas the second and third terms within each of the brackets in the second and third lines in Eq. (3.38) provide correction terms to the locally stationary approximation to the conditional response spectrum. These second and third terms

within the brackets result from the terms k = 1 and 2 in the expansion of $a_n(t-u)$ given by Eq. (3.33). Thus, the locally stationary response approximation to $\Phi_y(f,t|\sigma_f)$ that results from the locally stationary approximation to the turbulence spectrum $\Phi_{w_f}(f,t|\sigma_f)$ is

$$\Phi_y(f,t|\sigma_f) \approx \left| \Phi_{w_s}(f) + \sigma_f^2(t)\, \Phi_z(f) \right| \, |H(f)|^2 \quad , \tag{3.40}$$

which, of course, could have been written directly from the turbulence model of Eq. (3.1).

Equation (3.38), together with the above comments, provides us with the criteria that must be met for the approximation of Eq. (3.40) to be valid. For the locally stationary approximation to the *turbulence spectrum* $\Phi_{w_f}(f,t|\sigma_f)$ to be valid, the first term in the third line of Eq. (3.38) must be small in comparison with the first term in the second line. When Eq. (3.18b) is introduced into Eq. (3.38), *this locally stationary excitation condition becomes*

$$\left| \frac{d^2 \ln \sigma_f^2(t)}{dt^2} \right| << 32\pi^2 \; \frac{\Phi_z(f)\; |H(f)|^2}{|\Phi_z^{(2)}(f)|\; |H(f)|^2} \quad , \tag{3.41}$$

which must be satisfied in regions where $\sigma_f^2(t)$ is not negligible. Equation (3.41) is identical with Eq. (3.20), as expected, except for the terms $|H(f)|^2$ in the right-hand side of Eq. (3.41). Inclusion of the terms $|H(f)|^2$ relaxes the requirement of Eq. (3.20) to the extent that the right-hand side of Eq. (3.20) need be large in comparison with the left-hand side only over the range of frequencies where $|H(f)|^2$ is not negligible.

When the approximation of Eq. (3.41) is satisfied, our locally stationary response condition is satisfied if the second and third terms within the brackets in the second line of Eq. (3.38) are small in comparison with the first term. Using the fact that

$$\frac{d}{dt} \ln \sigma_f^2(t) = \frac{\frac{d}{dt}\, \sigma_f^2(t)}{\sigma_f^2(t)} \tag{3.42}$$

and using Eq. (3.39), *the first of our locally stationary response conditions becomes*

$$\left|\frac{d}{dt}\ \ell n\sigma_f^2(t)\right| << \frac{\Phi_z(f)\ |m_h^{(0)}(f)|}{\Phi_z(f)\ |m_h^{(1)}(f)|} \tag{3.43}$$

Notice that the conditions of Eqs. (3.41) and (3.43) both involve derivatives of $\ell n\sigma_f^2(t)$.

To relate the second locally stationary response condition to the derivatives of $\ell n\sigma_f^2(t)$, we note that

$$\frac{d}{dt}\ \ell n\sigma_f^2(t) = (\sigma_f^2)^{-1}\ \frac{d}{dt}\ \sigma_f^2$$

and

$$\frac{d^2}{dt^2}\ \ell n\sigma_f^2(t) = (\sigma_f^2)^{-1}\ \frac{d^2}{dt^2}\ \sigma_f^2 - (\sigma_f^2)^{-2}\left(\frac{d}{dt}\ \sigma_f^2\right)^2$$

$$= \frac{1}{\sigma_f^2}\ \frac{d^2}{dt^2}\ \sigma_f^2 - \left(\frac{d}{dt}\ \ell n\sigma_f^2\right)^2 \quad ; \tag{3.44}$$

hence, rearranging Eq. (3.44), we have

$$\frac{\dfrac{d^2}{dt^2}\ \sigma_f^2(t)}{\sigma_f^2(t)} = \frac{d^2}{dt^2}\ \ell n\sigma_f^2(t) + \left[\frac{d}{dt}\ \ell n\sigma_f^2(t)\right]^2 \quad . \tag{3.45}$$

From Eqs. (3.39) and (3.45), it is evident that the condition that the second term within the brackets in the second line of Eq. (3.38) be negligible in comparison with the first term may be expressed as

$$\left|\frac{d^2}{dt^2}\ \ell n\sigma_f^2(t) + \left[\frac{d}{dt}\ \ell n\sigma_f^2(t)\right]^2\right| << 2\ \frac{\Phi_z(f)\ |m_h^{(0)}(f)|}{\Phi_z(f)\ |m_h^{(2)}(f)|}\ , \tag{3.46}$$

which is *the second of our locally stationary response conditions. When Eqs. (3.41), (3.43), and (3.46) are all satisfied, the locally stationary conditional instantaneous response spectrum approximation given by Eq. (3.40) is valid.*

The physical significance of the conditions of Eqs. (3.41), (3.43), and (3.46) will be discussed in a later section of this report, where expressions for $m_h^{(1)}(f)$ and $m_h^{(2)}(f)$ will be given in terms of the magnitude and phase of the aircraft frequency response function H(f). Methods for evaluation of the required turbulence metrics to test for the satisfaction of the conditions of Eqs. (3.41), (3.43), and (3.46) also will be derived later in this report.

The importance of the locally stationary turbulence and response approximations has been pointed out by Sidwell – e.g., pp. 19-22 and p. 41 of Ref. 5.

AIRCRAFT RESPONSE EXCEEDANCE RATES AND PROBABILITY DENSITY FUNCTIONS

Gaussian Property of Response Process Conditioned on the $\sigma_f(t)$ Process

In this section, the locally stationary instantaneous response spectrum approximation given by Eq. (3.40) will be used to derive expressions for aircraft-response exceedance rates and probability density functions. In evaluating the usefulness of these results, we recall that the unit impulse-response function h(t), introduced in the last section, may be regarded as describing the response of any particular part of the aircraft modeled as a linear constant parameter structure. For example, h(t) may be chosen to describe the stress response at a critical section in a wing spar.

In the derivations to follow, we shall assume that the "slow" component $w_s(t)$ and the component z(t) in the turbulence model of Eq. (2.3) are both stationary Gaussian processes with zero mean values. *When $w_s(t)$ and $z(t)$ satisfy this stationary Gaussian assumption, the response process $y(t)$, conditioned on the process $\sigma_f(t)$, is a zero mean strictly Gaussian (generally nonstationary) process.* To prove this, we first note that each sample function of y(t) can be expressed as

$$y(t)|\sigma_f = y_s(t) + y_f(t)|\sigma_f \quad , \tag{4.1}$$

where

$$y_s(t) = \int_{-\infty}^{\infty} h(\tau)\ w_s(t-\tau)d\tau \quad , \tag{4.2}$$

and

$$y_f(t)|\sigma_f = \int_{-\infty}^{\infty} h(\tau)\ w_f(t-\tau)|\sigma_f\ d\tau \quad , \tag{4.3}$$

where the vertical bars followed by σ_f denote the fact that, in the stochastic process interpretation of the above results, the sample functions $\sigma_f(t)$ are to be regarded as known or specified but otherwise unrestricted functions of time. According to Eq. (4.1), $\{y(t)|\sigma_f\}$ is the sum of two (independent) processes $\{y_s(t)\}$ and $\{y_f(t)|\sigma_f\}$, since $w_s(t)$ and z(t) are assumed to be

independent. By assumption, $\{w_s(t)\}$ is a Gaussian process. Since any *linear* transformation of a Gaussian process results in a Gaussian process, e.g., see Cramer [*9*], pp. 312,313, it follows that $\{y_s(t)\}$ is a Gaussian process. Thus, since the sum of any number of independent Gaussian processes is itself Gaussian, e.g., Cramer [*9*], pp. 316-317, it follows that the process $\{y(t)|\sigma_f\}$ will be Gaussian (but generally nonstationary), if the process $\{y_f(t)|\sigma_f\}$ is Gaussian. However, according to Eq. (4.3), $\{y_f(t)|\sigma_f\}$ is a linear transformation of $\{w_f(t)|\sigma_f\}$; thus, $\{y(t)|\sigma_f\}$ will be Gaussian if $\{w_f(t)|\sigma_f\}$ is Gaussian. Furthermore, according to Eq. (2.4), when $\sigma_f(t)$ is considered as a known or specified function, $w_f(t)$ is a (memoryless) linear transformation of $z(t)$; hence, the process $\{w_f(t)|\sigma_f\}$ is Gaussian. Therefore, the conditional *response* process $\{y(t)|\sigma_f\}$ is a (generally nonstationary) Gaussian process. *This result does not depend on any of the locally stationary requirements of Eqs. (3.41), (3.43), or (3.46).*

In the following work, we shall require the first-order probability density functions of $\{y(t)\}$ conditioned on the process $\sigma_f(t)$. According to the above comments, this probability density $p(y|\sigma_f^2)$ is Gaussian; i.e.,*

$$p(y|\sigma_f^2) = \frac{1}{\sqrt{2\pi(\sigma_y^2|\sigma_f^2)}}\, e^{-\frac{y^2}{2(\sigma_y^2|\sigma_f^2)}} \tag{4.4}$$

where $(\sigma_y^2|\sigma_f^2)^{1/2}$ is the standard deviation of the response process $\{y(t)|\sigma_f\}$ given that $\sigma_f(t)$ is specified; i.e., $(\sigma_y^2|\sigma_f^2)^{1/2}$ is the square root of the conditional variance:

$$\sigma_y^2|\sigma_f^2 = E\{y^2|\sigma_f^2\} \quad . \tag{4.5}$$

We also shall require the fourth moment of $y(t)$, given that $\sigma_f(t)$ is specified. From Eq. (4.4) and known properties of the Gaussian density function – e.g., pp. 220-221 of Parzen [*10*] – this fourth moment is

$$E\{y^4|\sigma_f^2\} = 3(\sigma_y^2|\sigma_f^2)^2 \quad . \tag{4.6}$$

*We have used both $p(\cdots|\sigma_f)$ and $p(\cdots|\sigma_f^2)$ to denote the fact that the value of the process σ_f is specified. These are equivalent concepts. This equivalence is used in other notations as well.

General Expression for Aircraft Response Exceedance Rates

Numerous references in the aeronautical literature have dealt with the mean rate of exceedances of an aircraft-response variable y(t) past some specified threshold level. Here, we shall deal with the expected number of crossings *with positive slope* $N_+(y)$ of the response past a specified level y. It was shown by Rice [*11*] on pp. 189-193 of the Wax edition that, *for stationary processes,* one has

$$N_+(y) = \int_0^\infty \dot{y} p(y,\dot{y}) d\dot{y} \quad , \tag{4.7}$$

where $p(y,\dot{y})$ is the joint probability density function of y and its derivative $\dot{y}$. A derivation of this result also is given on pp. 45-47 of Crandall and Mark [*12*]. The result of Eq. (4.7) applies to nonGaussian as well as Gaussian processes. Using the conditional joint probability density function $p(y,\dot{y}|\sigma_f^2)$ of the aircraft response y and its derivative $\dot{y}$, we may formally express Eq. (4.7) as

$$\begin{aligned} N_+(y) &= \int_0^\infty \dot{y} \int_0^\infty p(y,\dot{y}|\sigma_f^2)\, p(\sigma_f^2) d\sigma_f^2\, d\dot{y} \\ &= \int_0^\infty \left[\int_0^\infty \dot{y}\, p(y,\dot{y}|\sigma_f^2) d\dot{y} \right] p(\sigma_f^2) d\sigma_f^2 \\ &= \int_0^\infty N_+(y|\sigma_f^2)\, p(\sigma_f^2) d\sigma_f^2 \quad , \end{aligned} \tag{4.8}$$

where we have defined

$$N_+(y|\sigma_f^2) = \int_0^\infty \dot{y}\, p(y,\dot{y}|\sigma_f^2) d\dot{y} \quad , \tag{4.9}$$

which we may regard as the expected number of threshold crossings with positive slope of the aircraft response past the specified level y, given that $\sigma_f^2(t)$ is specified.

Several comments about Eqs. (4.8) and (4.9) are in order at this juncture. First, $p(y,\dot{y}|\sigma_f^2)$ designates the joint probability density of the response y and its derivative $\dot{y}$, given that the stochastic *function* $\sigma_f^2(t)$ is specified. Therefore, $p(y,\dot{y}|\sigma_f^2)$ is a function of time. Furthermore, because of this fact, the integration involved in the formal determination of $p(y,\dot{y})$ from $p(y,\dot{y}|\sigma_f^2)$ – i.e.,

$$p(y,\dot{y}) = \int_0^\infty p(y,\dot{y}|\sigma_f^2)\, p(\sigma_f^2)d\sigma_f^2 \quad , \tag{4.10}$$

– must actually be regarded as an infinite dimensional integration over an infinite dimensional space $\sigma_f^2(t_1)$, $\sigma_f^2(t_2)$, $\sigma_f^2(t_3)$, ... as $t_{i+1}-t_i$, for all $i = 1,2,3,...$ is made to shrink to zero. However, when interpreted in this manner, the operations in Eqs. (4.8) and (4.9) are valid. In particular, it follows from the above derivation that Eq. (4.9) is valid even though the conditional process $\{y(t)|\sigma_f(t)\}$ is not stationary.

In order to evaluate $p(y,\dot{y}|\sigma_f^2)$ for the purposes of this report, we shall want to assume that $\{y(t)|\sigma_f(t)\}$ is locally stationary. When $\{y(t)|\sigma_f(t)\}$ is stationary with zero mean value, it is known that $E\{y\,\dot{y}|\sigma_f\} = 0$; i.e., y and $\dot{y}$ are uncorrelated. It is shown in Appendix A of this report that when the conditional instantaneous spectrum of the aircraft response satisfies Eq. (3.40) – i.e., when the conditions of Eqs. (3.41), (3.43), and (3.46) are satisfied – the conditional correlation coefficient

$$\rho_{y\dot{y}}|\sigma_f^2 = \frac{E\{y\dot{y}|\sigma_f^2\}}{\sqrt{(\sigma_y^2|\sigma_f^2)(\sigma_{\dot{y}}^2|\sigma_f^2)}} \tag{4.11}$$

is negligibly small. Hence, for cases where Eq. (3.40) is valid and where the turbulence processes $\{w_S(t)\}$ and $\{z(t)\}$ in Eq. (2.3) are Gaussian with zero mean values, the joint conditional probability density $p(y,\dot{y}|\sigma_f^2)$ is two-dimensional Gaussian in the uncorrelated variates y and $\dot{y}$. Hence, *when the locally stationary condition of Eq. (3.40) is satisfied,* we may apply Rice's well known formula [*11*], p. 193 of the Wax edition, to the evaluation of Eq. (4.9):

$$N_+(y|\sigma_f^2) = \frac{1}{2\pi}\left[-\frac{\phi_y''(0,t|\sigma_f)}{\phi_y(0,t|\sigma_f)}\right]^{\frac{1}{2}} e^{-\frac{y^2}{2\sigma_y^2|\sigma_f^2}} \quad , \tag{4.12}$$

where $\phi_y(\tau,t|\sigma_f)$ is the conditional instantaneous autocorrelation function of the response, defined in a manner completely analogous to Eq. (3.2a), and where the differentiation indicated in the brackets is with respect to τ.

To evaluate the quantity within the brackets in Eq. (4.12), we first note from the Fourier mate to Eq. (3.5) that the conditional instantaneous autocorrelation function $\phi_y(\tau,t|\sigma_f)$ is the inverse Fourier transform with respect to f of the conditional instantaneous spectrum $\Phi_y(f,t|\sigma_f)$; i.e.,

$$\phi_y(\tau,t|\sigma_f) = \int_{-\infty}^{\infty} \Phi_y(f,t|\sigma_f)e^{i2\pi f\tau}\, df \quad . \tag{4.13}$$

Hence, setting τ equal to zero in Eq. (4.13) yields

$$\begin{aligned}\phi_y(0,t|\sigma_f) &= \int_{-\infty}^{\infty} \Phi_y(f,t|\sigma_f)df \\ &= \sigma_y^2(t)|\sigma_f^2 \quad .\end{aligned} \tag{4.14}$$

Furthermore, differentiation of Eq. (4.13) twice with respect to τ yields, after setting τ equal to zero in the resulting expression:

$$\begin{aligned}\phi_y''(0,t|\sigma_f) &= -(2\pi)^2 \int_{-\infty}^{\infty} f^2\Phi_y(f,t|\sigma_f)df \\ &= -\sigma_{\dot{y}}^2(t)|\sigma_f^2 \quad ,\end{aligned} \tag{4.15}$$

where the second line is a well-known result – e.g., pp. 190-192 of the Wax edition of Ref. 11, or pp. 47-48 of Ref. 12 – and where the validity of the second line depends on the locally stationary property of Eq. (3.40).

We may now use Eq. (3.40) to evaluate the right-hand sides of Eqs. (4.14) and (4.15). Combining Eqs. (3.40) and (4.14) yields

$$\sigma_y^2(t)|\sigma_f^2 = \sigma_{y_s}^2 + \sigma_f^2(t)\,\sigma_{y_z}^2 \quad , \tag{4.16}$$

where

$$\sigma_{y_s}^2 \triangleq \int_{-\infty}^{\infty} \Phi_{w_s}(f)\,|H(f)|^2 df \quad , \tag{4.17}$$

and

$$\sigma_{y_z}^2 \triangleq \int_{-\infty}^{\infty} \Phi_z(f)\,|H(f)|^2 df \quad . \tag{4.18}$$

Similarly, combining Eqs. (3.40) and (4.15) yields

$$\sigma_{\dot{y}}^2(t)|\sigma_f^2 = \sigma_{\dot{y}_s}^2 + \sigma_f^2(t)\sigma_{\dot{y}_z}^2 \quad , \tag{4.19}$$

where

$$\sigma_{\dot{y}_s}^2 = (2\pi)^2 \int_{-\infty}^{\infty} f^2\, \Phi_{w_s}(f)\,|H(f)|^2 df \quad , \tag{4.20}$$

and

$$\sigma_{\dot{y}_z}^2 = (2\pi)^2 \int_{-\infty}^{\infty} f^2\, \Phi_z(f)\,|H(f)|^2 df \quad . \tag{4.21}$$

Finally, by combining Eqs. (4.14), (4.15), (4.16), and (4.19) with Eq. (4.12), we obtain our final expression for the conditional rate of exceedance crossings with positive slope:

$$N_+(y|\sigma_f^2) = \frac{1}{2\pi}\left[\frac{\sigma_{\dot{y}_s}^2 + \sigma_f^2(t)\sigma_{\dot{y}_z}^2}{\sigma_{y_s}^2 + \sigma_f^2(t)\sigma_{y_z}^2}\right]^{\frac{1}{2}} e^{-\frac{y^2}{2[\sigma_{y_s}^2+\sigma_f^2(t)\sigma_{y_z}^2]}} \quad , \tag{4.22}$$

which is valid whenever the locally stationary approximation of Eq. (3.40) is valid. The quantities $\sigma_{y_s}^2$, $\sigma_{y_z}^2$, $\sigma_{\dot{y}_s}^2$, and $\sigma_{\dot{y}_z}^2$

within the brackets in Eq. (4.22) are to be evaluated using Eqs. (4.17), (4.18), (4.20), and (4.21). When Eq. (4.22) is substituted into Eq. (4.8), we obtain a general expression for the mean exceedance rate of the aircraft response expressed in terms of the probability density function $p(\sigma_f^2)$ of the stochastic function $\sigma_f^2(t)$.

Series Approximation of Aircraft Response Exceedance Rates

Evaluation of the expression for the response exceedance rate $N_+(y)$ given by Eq. (4.8) requires knowledge of the probability density function $p(\sigma_f^2)$. However, for cases where the variance of σ_f^2 – i.e.,

$$\mu^{(2)}_{\sigma_f^2} \triangleq E\{[\sigma_f^2 - E(\sigma_f^2)]^2\}$$

$$= E\{[\sigma_f^2]^2\} - \{E[\sigma_f^2]\}^2 \tag{4.23}$$

– is small in comparison with the square of the mean $E\{\sigma_f^2\}$ of σ_f^2, a useful series approximation to $N_+(y)$ can be obtained, as we shall now show.

Let us consider $N_+(y|\sigma_f^2)$ as a function of the variable σ_f^2. Consider the Taylor's series expansion of $N_+(y|\sigma_f^2)$, where the expansion is to be centered about the expected value of σ_f^2 – i.e.,

$$\overline{\sigma_f^2} = E\{\sigma_f^2\} \quad . \tag{4.24}$$

We may formally write this expansion as

$$N_+(y|\sigma_f^2) = \sum_{k=0}^{\infty} \frac{1}{k!} N_+^{(k)}(y|\overline{\sigma_f^2}) \; (\sigma_f^2-\overline{\sigma_f^2})^k \tag{4.25a}$$

$$= N_+(y|\overline{\sigma_f^2}) + N_+^{(1)}(y|\overline{\sigma_f^2}) \; (\sigma_f^2-\overline{\sigma_f^2})$$

$$+ \frac{1}{2} N_+^{(2)}(y|\overline{\sigma_f^2}) \; (\sigma_f^2-\overline{\sigma_f^2})^2 + \ldots \quad , \tag{4.25b}$$

where we have defined

$$N_+^{(k)}(y|\overline{\sigma_f^2}) \triangleq \left.\frac{d^k N_+(y|\sigma_f^2)}{d(\sigma_f^2)^k}\right|_{\sigma_f^2 = \overline{\sigma_f^2}} \qquad (4.26)$$

and

$$N_+^{(0)}(y|\overline{\sigma_f^2}) \triangleq N_+(y|\sigma_f^2)\Big|_{\sigma_f^2 = \overline{\sigma_f^2}} \quad . \qquad (4.27)$$

Let us now substitute Eq. (4.25a) into Eq. (4.8) and interchange the orders of integration and summation:

$$N_+(y) = \sum_{k=0}^{\infty} \frac{1}{k!} N_+^{(k)}(y|\overline{\sigma_f^2}) \int_0^{\infty} (\sigma_f^2-\overline{\sigma_f^2})^k \; p(\sigma_f^2)d(\sigma_f^2)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} N_+^{(k)}(y|\overline{\sigma_f^2})\mu_{\sigma_f^2}^{(k)} \quad , \qquad (4.28)$$

where

$$\mu_{\sigma_f^2}^{(k)} \triangleq \int_0^{\infty} (\sigma_f^2-\overline{\sigma_f^2})^k \; p(\sigma_f^2)d(\sigma_f^2) \qquad (4.29)$$

is the *kth* central moment of the stochastic variable σ_f^2. The reason for choosing the expansion center of the Taylor's series of Eq. (4.25) about the point $\sigma_f^2 = \overline{\sigma_f^2}$ may be seen from Eqs. (4.28) and (4.29). According to Eq. (4.29),

$$\mu_{\sigma_f^2}^{(1)} \equiv 0 \quad ; \qquad (4.30)$$

i.e., the term k = 1 in the expansion of Eq. (4.28) vanishes when the center of the Taylor's series expansion is $\overline{\sigma_f^2}$. Furthermore, it is known - e.g., Ref. 9, p. 175 - that the second moment of σ_f^2 is minimized when taken about $\overline{\sigma_f^2}$; consequently, the term k=2 in the expansion of Eq. (4.28) is minimized when the expansion center of the Taylor's series expansion of Eq. (4.25) is taken as $\overline{\sigma_f^2}$. The first few terms of Eq. (4.28) are

$$N_+(y) = N_+(y|\overline{\sigma_f^2}) + \frac{1}{2}\, \mu^{(2)}_{\sigma_f^2}\, N_+^{(2)}(y|\overline{\sigma_f^2}) + \dots \quad . \tag{4.31}$$

Discussion. It is immediately obvious using conditional expectations – e.g., Ref. 8, pp. 55-56 – that when we consider σ_f^2 to be a stochastic variable, we may express the mean-square aircraft response $E\{y^2\}$ as

$$E\{y^2\} = \int_0^\infty E\{y^2|\sigma_f^2\}\; p(\sigma_f^2) d\sigma_f^2$$

$$= \int_0^\infty [\sigma^2_{y_s} + \sigma_f^2 \sigma^2_{y_z}]\; p(\sigma_f^2) d\sigma_f^2$$

$$= \sigma^2_{y_s} + \overline{\sigma_f^2}\; \sigma^2_{y_z} \quad , \tag{4.32}$$

where we have used Eq. (4.16), with an obvious minor change in notation, and the definition of Eq. (4.24). Consequently, it follows directly from Eqs. (4.22) and (4.32) that *the first term $N_+(y|\overline{\sigma_f^2})$ in the series of Eq. (4.31) is the response exceedance rate one would compute by assuming that the aircraft response $y(t)$ is a stationary Gaussian process with a variance equal to the actual variance $E\{y^2\}$ given by Eq. (4.32). It follows that the terms $k = 2,3,\dots$ in Eqs. (4.28) and (4.31) are correction terms to the Gaussian approximation $N_+(y|\overline{\sigma_f^2})$ of the true aircraft exceedance rate $N_+(y)$, where these correction terms take into account the nonGaussian nature of the aircraft response that is caused by the stochastic variations in the quantity $\sigma_f(t)$ in the model of Eq. (2.3).*

The behavior of the first correction term,

$$\frac{1}{2}\, N_+^{(2)}(y|\overline{\sigma_f^2})\; \mu^{(2)}_{\sigma_f^2} \quad ,$$

to the Gaussian approximation $N_+(y|\overline{\sigma_f^2})$ in Eq. (4.31) is of special interest, since this correction term governs the behavior of the deviation from Gaussian behavior $N_+(y)$ when $\mu^{(2)}_{\sigma_f^2}$ is small in comparison with $(\overline{\sigma_f^2})^2$. It is shown in Appendix B of this

report that $N_+^{(2)}(y|\overline{\sigma_f^2})$ can be expressed, exactly, as

$$N_+^{(2)}(y|\overline{\sigma_f^2}) = 2\, N_+(y|\overline{\sigma_f^2})\, Q^{(2)}(y|\overline{\sigma_f^2}) \quad , \tag{4.33}$$

where

$$Q^{(2)}(y|\sigma_f^2) = \frac{1}{4}\left\{\frac{1}{2}\left[\frac{\sigma^2_{y_z}}{\sigma_y^2|\sigma_f^2}\left(\frac{y^2}{\sigma_y^2|\sigma_f^2} - 1\right) + \frac{\sigma^2_{\dot{y}_z}}{\sigma_{\dot{y}}^2|\sigma_f^2}\right]^2 - \left[\frac{(\sigma^2_{y_z})^2}{(\sigma_y^2|\sigma_f^2)^2}\left(2\,\frac{y^2}{\sigma_y^2|\sigma_f^2} - 1\right) + \frac{(\sigma^2_{\dot{y}_z})^2}{(\sigma_{\dot{y}}^2|\sigma_f^2)^2}\right]\right\} \tag{4.34}$$

and where we have used the notation

$$\sigma_y^2|\sigma_f^2 = \sigma^2_{y_s} + \sigma_f^2\sigma^2_{y_z} \tag{4.35a}$$

and

$$\sigma_{\dot{y}}^2|\sigma_f^2 = \sigma^2_{\dot{y}_s} + \sigma_f^2\sigma^2_{\dot{y}_z} \quad . \tag{4.35b}$$

When Eq. (4.33) is substituted into Eq. (4.31), and only one correction term to the Gaussian approximation is retained, we obtain

$$N_+(y) \simeq N_+(y|\overline{\sigma_f^2})\,[1 + \mu^{(2)}_{\sigma_f^2}\, Q^{(2)}(y|\overline{\sigma_f^2})] \quad ; \tag{4.36}$$

hence, we see that the quantity $Q^{(2)}(y|\overline{\sigma_f^2})$ is of the nature of a multiplicative correction factor to the Gaussian approximation $N_+(y|\overline{\sigma_f^2})$.

Often – e.g., Ref. 13 – the logarithm of the normalized exceedance rate $N_+(y)/N_+(0)$ is plotted. Dividing Eq. (4.36) by $N_+(0)$ and taking the natural logarithm of the resulting expression gives

$$\ln \frac{N_+(y)}{N_+(0)} = \ln \frac{N_+(y|\overline{\sigma_f^2})}{N_+(0)} + \ln[1 + \mu^{(2)}_{\sigma_f^2} Q^{(2)}(y|\overline{\sigma_f^2})]$$

$$\simeq \ln \frac{N_+(y|\overline{\sigma_f^2})}{N_+(0)} + \mu^{(2)}_{\sigma_f^2}\, Q^{(2)}(y|\overline{\sigma_f^2}) \quad , \tag{4.37}$$

$$\text{for } \mu^{(2)}_{\sigma_f^2}\, Q^{(2)}(y|\overline{\sigma_f^2}) \leq 0.2 \quad ,$$

where, in going to the second line, we have used the first term in the Maclaurin expansion of $\ln(1+x)$ — i.e., $\ln(1+x) \approx x$, which has approximately ten percent error when $x = 0.2$. *It follows from Eq. (4.37) that, for sufficiently small values of* $\mu^{(2)}_{\sigma_f^2}$*, the function* $Q^{(2)}(y|\overline{\sigma_f^2})$ *governs the deviation of a plot of* $\ln[N_+(y)/N_+(0)]$ *from the quantity* $\ln[N_+(y|\overline{\sigma_f^2})/N_+(0)]$ *which is, except for a possible constant factor, the result predicted if one assumes that* $y(t)$ *is stationary and Gaussian.*

In order to discuss further the result of Eq. (4.37), we shall restrict our attention to the case where the aircraft response to the component $w_s(t)$ of our model of Eq. (2.3) is negligible. This assumption implies that $\sigma^2_{y_s} \equiv 0$ in Eqs. (4.35a) and (4.35b). Consequently, for this limiting case, we obtain from Eqs. (4.35a) and (4.35b) *when* $\sigma^2_{y_s}$ *and* $\sigma^2_{\dot{y}_s}$ *are set equal to zero,*

$$\frac{\sigma^2_{y_z}}{\sigma^2_y|\sigma^2_f} = \frac{1}{\sigma^2_f} \quad , \quad \frac{\sigma^2_{\dot{y}_z}}{\sigma^2_{\dot{y}}|\sigma^2_f} = \frac{1}{\sigma^2_f} \quad . \tag{4.38a,b}$$

Substituting Eqs. (4.38a,b) into Eq. (4.34) and writing σ^2_y for $\sigma^2_y|\sigma^2_f$, we obtain, after simplification,

$$Q^{(2)}(y|\overline{\sigma_f^2}) = \frac{1}{8(\overline{\sigma_f^2})^2} \frac{y^2}{\sigma^2_y} \left(\frac{y^2}{\sigma^2_y} - 4 \right) , \tag{4.39}$$

which is valid only when $\sigma_{y_s} = 0$ *and* $\sigma_{\dot{y}_s} = 0$. Notice that, for this case, $\sigma^2_y = E(\sigma^2_f)\sigma^2_{y_z}$; hence, the quantity σ^2_y in Eq. (4.39) is the true variance of the response variate $y(t)$. Furthermore,

according to Eq. (4.39), we have $Q^{(2)}(0|\overline{\sigma_f^2}) = 0$. Hence, according to the approximation of Eq. (4.36), we have $N_+(0) = N_+(0|\overline{\sigma_f^2})$. It follows that, for the case under consideration, the first term in the right-hand side of Eq. (4.37) can be written as $\ln[N_+(y|\overline{\sigma_f^2})/N_+(0|\overline{\sigma_f^2})]$ which is, exactly, the logarithm of the normalized exceedance rate for a stationary Gaussian process. Using Eq. (4.22) to evaluate this term and writing σ_y^2 for $\overline{\sigma_f^2}\sigma_{y_z}^2$ (recall that $\sigma_{y_s}^2 = 0$), we have for the right-hand side of Eq. (4.37),

$$\ln \frac{N_+(y)}{N_+(0)} \simeq -\frac{y^2}{2\sigma_y^2} + \frac{1}{8}\,\frac{\mu^{(2)}_{\sigma_f^2}}{(\overline{\sigma_f^2})^2}\,\frac{y^2}{\sigma_y^2}\left(\frac{y^2}{\sigma_y^2} - 4\right) \quad , \tag{4.40}$$

where we have used Eq. (4.39) and where, according to Eq. (4.37), the correction term, which is the second term on the right-hand side of Eq. (4.40), should be accurate to within about ten percent whenever that term is less than about 0.2.

Equation (4.40) is the desired result. If $\ln[N_+(y)/N_+(0)]$ is plotted on the ordinate versus y^2/σ_y^2 on the abscissa, we see that the first term in the right-hand side of Eq. (4.40) is a straight line with slope of minus one-half and ordinate intersection at $(y^2/\sigma_y^2) = 0$. This first term is the exact result for stationary Gaussian processes $y(t)$. The second term on the right-hand side of Eq. (4.40) is a parabola in the quantity y^2/σ_y^2. This second term is zero at $(y^2/\sigma_y^2) = 0$ and at $(y^2/\sigma_y^2) = 4$. Between these two values, this correction term is negative (since $\mu^{(2)}_{\sigma_f^2}$ is necessarily positive or zero). For values of $(y^2/\sigma_y^2) > 4$, this correction term is positive. *Consequently, for threshold levels between $0 < |y| < 2\sigma_y$, Eq. (4.40) indicates that the first (stationary Gaussian) term overestimates the number of threshold crossings; whereas, for thresholds $|y| > 2\sigma_y$, Eq. (4.40) indicates that the first (stationary Gaussian) term underestimates the number of threshold crossings of processes $y(t)$ with time-varying variance.**

Equation (4.40) provides a theoretical prediction of the concave shape of aircraft-response logarithmic exceedance plots for the turbulence model described by Eq. (2.3) for the case where $w_s(t) = 0$. The "strength" of the parabolic correction (in the variable y^2/σ_y^2) to the stationary Gaussian response result, given by the first term on the right-hand side, is governed by the coefficient

*For $y = 0$, it is evident from Eqs. (4.36) and (4.39) that the stationary Gaussian term gives the correct value for $N_+(0)$ since $Q^{(2)}(0|\overline{\sigma_f^2})$ is zero.

$$\frac{\mu^{(2)}_{\sigma_f^2}}{(\overline{\sigma_f^2})^2} = \frac{E\{[\sigma_f^2 - E(\sigma_f^2)]^2\}}{\{E[\sigma_f^2]\}^2} \tag{4.41a}$$

$$= \frac{E\{[\sigma_f^2]^2\} - \{E[\sigma_f^2]\}^2}{E\{[\sigma_f^2]\}^2} \quad , \tag{4.41b}$$

which is the square of the *coefficient of variation* (e.g., Ref. 9) of the time-varying variance $\sigma_f^2(t)$ in the turbulence model of Eq. (2.3).

Aircraft Response Probability Density Functions

It is evident from Eq. (4.12) that, when the aircraft response $y(t)$ is a stationary Gaussian process, the mean rate of "up crossings" $N_+(y)$ of the level y is proportional to the probability density function (pdf) $p(y)$ of $y(t)$. We shall show in this section that the proportionality between $N_+(y)$ and $p(y)$ is no longer maintained when the turbulence excitation is modeled by Eq. (2.3).

First, we shall derive a series expansion for the first order pdf $p(y)$ of the response process $\{y(t)\}$ using a method completely analogous to that used to obtain our series representation of $N_+(y)$. Denoting by $p(y|\sigma_f^2)$ the conditional pdf of y given that σ_f^2 is specified, we have

$$p(y) = \int_0^\infty p(y|\sigma_f^2)\; p(\sigma_f^2) d\sigma_f^2 \quad . \tag{4.42}$$

We now formally expand $p(y|\sigma_f^2)$ in a Taylor's series in the variable σ_f^2 about the expansion center $E\{\sigma_f^2\} = \overline{\sigma_f^2}$:

$$p(y|\sigma_f^2) = \sum_{k=0}^{\infty} \frac{1}{k!}\, p^{(k)}(y|\overline{\sigma_f^2})\; (\sigma_f^2 - \overline{\sigma_f^2})^k \quad , \tag{4.43}$$

where we have defined

$$p^{(k)}(y|\overline{\sigma_f^2}) \triangleq \left.\frac{d^k p(y|\sigma_f^2)}{d(\sigma_f^2)^k}\right|_{\sigma_f^2 = \overline{\sigma_f^2}} \tag{4.44}$$

and

$$p^{(0)}(y|\overline{\sigma_f^2}) = p(y|\sigma_f^2)\big|_{\sigma_f^2 = \overline{\sigma_f^2}} \quad . \tag{4.45}$$

Next, we substitute Eq. (4.43) into Eq. (4.42) and interchange the orders of integration and summation, finding thereby that

$$p(y) = \sum_{k=0}^{\infty} \frac{1}{k!}\, p^{(k)}(y|\overline{\sigma_f^2})\, \mu_{\sigma_f^2}^{(k)} \quad , \tag{4.46}$$

where we have used the definition of Eq. (4.29). The motivation for using $\overline{\sigma_f^2}$ as the expansion center of the Taylor's series expansion of Eq. (4.43) is the same as that described earlier in the text between Eqs. (4.30) and (4.31). We may write out the first few terms of Eq. (4.46) as

$$p(y) = p(y|\overline{\sigma_f^2}) + \frac{1}{2}\, \mu_{\sigma_f^2}^{(2)}\, p^{(2)}(y|\overline{\sigma_f^2}) + \cdots \quad , \tag{4.47}$$

since the term in Eq. (4.46) corresponding to $k = 1$ is identically zero because we have chosen our Taylor's series expansion center at $\sigma_f^2 = \overline{\sigma_f^2}$.

The first term in the right-hand side of Eq. (4.47) is the Gaussian approximation to the pdf of $y(t)$ given by Eq. (4.4) with $\sigma_f^2 = \overline{\sigma_f^2}$; hence, according to Eq. (4.35a), we have

$$\sigma_y^2|\overline{\sigma_f^2} = \sigma_{y_s}^2 + \overline{\sigma_f^2}\sigma_{y_z}^2 = E\{y^2\} = \sigma_y^2 \quad . \tag{4.48}$$

The second term in the right-hand side of Eq. (4.47) is the "first-order" correction term to the Gaussian approximation provided by the first term. When the coefficient of variation $[\mu_{\sigma_f^2}^{(2)}/(\overline{\sigma_f^2})^2]^{1/2}$ of the random variable σ_f^2 is sufficiently small, this first-order correction term will exhibit the principal deviation of $p(y)$ from the Gaussian approximation $p(y|\overline{\sigma_f^2})$.

It is shown in Appendix C of this report that $p^{(2)}(y|\overline{\sigma_f^2})$ can be expressed, exactly, as

$$p^{(2)}(y|\overline{\sigma_f^2}) = 2\,p(y|\overline{\sigma_f^2})\,U^{(2)}(y|\overline{\sigma_f^2}) \quad , \tag{4.49}$$

where

$$U^{(2)}(y|\sigma_f^2) = \frac{1}{4}\,\frac{(\sigma_{y_z}^2)^2}{(\sigma_y^2|\sigma_f^2)^2}\left[1 - \frac{2y^2}{\sigma_y^2|\sigma_f^2} + \frac{1}{2}\left(\frac{y^2}{\sigma_y^2|\sigma_f^2} - 1\right)^2\right]$$

$$= \frac{1}{8}\,\frac{(\sigma_{y_z}^2)^2}{(\sigma_y^2|\sigma_f^2)^2}\left[\left(\frac{y^2}{\sigma_y^2|\sigma_f^2}\right)^2 - 6\,\frac{y^2}{\sigma_y^2|\sigma_f^2} + 3\right] \quad . \tag{4.50}$$

Substitution of Eq. (4.49) into Eq. (4.47) yields the two-term approximation to $p(y)$:

$$p(y) \approx p(y|\overline{\sigma_f^2})\left[1 + \mu_{\sigma_f^2}^{(2)}U^{(2)}(y|\overline{\sigma_f^2})\right] \quad . \tag{4.51}$$

Discussion. It is in the nature of the derivation of Eqs. (4.46), (4.47), and (4.51) that, when $\mu_{\sigma_f^2}^{(2)}$ is sufficiently small, the right-hand side of Eq. (4.51) should provide a good approximation to $p(y)$. It is of considerable interest that, using an apparently completely different line of reasoning – namely, the Gram-Charlier series – e.g., Cramer, Ref. 9 – one obtains precisely the same correction to the Gaussian approximation given by Eq. (4.51). This equivalence is most readily seen by a comparison of Eqs. (4.50) and (4.51) above with Eq. (8-111) on p. 272 of Papoulis [*14*], where we note that the quantity within the brackets in Eq. (4.50) above is the Hermite polynomial of degree four in the variable $y/(\sigma_y^2|\sigma_f^2)^{1/2}$.

To illustrate the equivalence of Eq. (4.51) to the Gram-Charlier series with the same number of terms, we must show, by comparing Eqs. (4.50) and (4.51) above with Eq. (8-111) of Ref. 14, or with Eqs. (17.6.5) and (17.6.6) of Ref. 9, that the coefficient of excess $\gamma_y^{(2)}$ of y – i.e.,

$$\gamma_y^{(2)} = \frac{\mu_y^{(4)}}{(\sigma_y^2)^2} - 3 \tag{4.52}$$

(where $\mu_y^{(4)}$ is the fourth central moment of y) – is related to $\mu_{\sigma_f}^{(2)}$ by

$$\frac{\gamma_y^{(2)}}{4!} = \frac{\mu_{\sigma_f^2}^{(2)}}{8} \; \frac{(\sigma_{y_z}^2)^2}{(\sigma_y^2|\overline{\sigma_f^2})^2} \tag{4.53}$$

or, equivalently, that

$$\gamma_y^{(2)} = 3\; \mu_{\sigma_f^2}^{(2)} \; \frac{(\sigma_{y_z}^2)^2}{(\sigma_y^2|\overline{\sigma_f^2})^2} \quad . \tag{4.54}$$

The equality in Eq. (4.54) is proved in Appendix D.

The main purpose of the above discussions of response exceedance rates and probability density functions has been to illustrate the importance of the pdf $p(\sigma_f^2)$ of the time-varying variance $\sigma_f^2(t)$. The fundamental importance of $p(\sigma_f^2)$ in aircraft-response statistics is clearly illustrated by Eqs. (4.8) and (4.42). Furthermore, it is evident from the series expansions of Eqs. (4.28) and (4.46) that the (central) moments $\mu_{\sigma_f^2}^{(k)}$, $k = 2,3,4,\ldots$ constitute the most important set of *parametric* descriptors of the pdf of σ_f^2. Methods for computing these quantities from turbulence records will be described in Sec. 6 of this report.

Use of conditional probabilities in computing an expression for exceedance rates analogous to Eq. (4.8) of this report, but using $p(\sigma_f)$ rather than $p(\sigma_f^2)$, has been used recently by Sidwell [*15*].

Comparison of Exceedance Rates and Probability Density Functions

It is evident from Eqs. (4.4), (4.12), (4.36), and (4.51) that, considered as a function of the response variable y, the Gaussian approximations to the exceedance rate $N_+(y)$ and the pdf p(y) provided by the first terms in the right-hand sides of Eqs. (4.36) and (4.51) are proportional to one another; i.e., each has the form $C \exp[-y^2/(2\sigma_y^2|\sigma_f^2)]$, where C is a constant.

However, this same statement is not true of the approximations to $N_+(y)$ and $p(y)$ provided by the right-hand sides of Eqs. (4.36) and (4.51), respectively. In particular, for the case where $\sigma_{y_s}^2 = 0$, it follows from Eqs. (4.36) and (4.39) that the zeros of the correction term $Q^{(2)}(y|\overline{\sigma_f^2})$ occur at $(y^2/\sigma_y^2) = 0, 4$; whereas, from Eqs. (4.50) and (4.51) it follows that the zeros of the correction term $U^{(2)}(y|\overline{\sigma_f^2})$ occur at $(y^2/\sigma_y^2) = (3 \pm \sqrt{6}) \approx 0.55, 5.45$. Thus, differences between the shapes of observed exceedance functions $N_+(y)$ and probability densities $p(y)$ are an indication of nonGaussian behavior of the turbulence excitation process $w(t)$.*

*On p. 34 of Ref. 5, Sidwell has pointed out that $N_+(y)$ and $p(y)$ will, in general, have different functional forms.

LOCALLY STATIONARY RESPONSE REQUIREMENT INTERPRETATIONS AND RELATIONSHIPS TO TURBULENCE MEASUREMENTS

Interpretation of Locally Stationary Instantaneous-Response-Spectrum Requirements

Three requirements for the validity of the approximation of Eq. (3.40) for computation of the conditional instantaneous aircraft response spectrum were arrived at in Sec. 3 of this report. These requirements are given by Eqs. (3.41), (3.43), and (3.46).

The first requirement, given by Eq. (3.41), has already been discussed in Ref. 6 in terms of a spatial variable $x = Vt$, where V is the aircraft speed. [See Eq. (4.82) of Ref. 5.] When the spectrum of $z(t)$ of the model of Eq. (2.3) is the von Karman transverse spectrum with integral scale L_z, the condition of Eq. (3.41) can be written as

$$\left|\frac{d^2 \ln\sigma_f^2(t)}{dt^2}\right| \leq 0.08\,\frac{V^2}{L_z^2} \quad , \tag{5.1}$$

where we note that $\ln\sigma^2 = 2\ln\sigma$. The condition of Eq. (5.1) means that the second derivative of $\ln\sigma_f^2(t)$, measured on the time scale $t' \equiv L_z/V$, must be less than or equal to 0.08. *This requirement means, roughly, that the relative changes in the time-varying variance $\sigma_f^2(t)$ of the "fast turbulence component" $w_f(t)$ in Eq. (2.3) must occur slowly when measured on the time scale $t' = L_z/V$ defined by the aircraft speed V and the integral scale L_z (i.e., nominal correlation interval) of the component z.* We refer the reader to the discussion of Eq. (4.82) in Ref. 6 for further interpretation of Eq. (5.1).

Equation (5.1) is independent of properties of the aircraft unit-impulse response function $h(t)$, while the second and third requirements are dependent on properties of $h(t)$. The second requirement given by Eq. (3.43), depends on the magnitude of the ratio $m_h^{(0)}(f)/m_h^{(1)}(f)$, where these two quantities are defined by Eq. (3.36). The quantities $m_h^{(k)}(f)$, $k = 0,1,2,\ldots$ were defined earlier in Ref. 16 by the author of this report for use in another context. It is shown on p. 282 of Ref. 16 that the quantity $m_h^{(1)}(f)/m_h^{(0)}(f)$ is the *group delay* $\tau_h(f)$ of the aircraft unit-impulse response $h(t)$; i.e.,

$$\tau_h(f) \triangleq \frac{m_h^{(1)}(f)}{m_h^{(0)}(f)} \quad . \tag{5.2}$$

If we express the Fourier transform, Eq. (3.31) of h(t) in terms of its magnitude and phase

$$H(f) = |H(f)| \, e^{i\theta_h(f)} \quad , \tag{5.3}$$

then $\tau_h(f)$ can be expressed in terms of the phase by

$$\tau_h(f) = \frac{m_h^{(1)}(f)}{m_h^{(0)}(f)} = -\frac{1}{2\pi}\frac{d\theta_h(f)}{df} \quad . \tag{5.4}$$

The group delay $\tau_h(f)$ of h(t) can be interpreted physically as an energy-spectrum-weighted frequency-decomposition of the time centroid $\overline{t_h}$ of $h^2(t)$; i.e.,

$$\overline{t_h} = \frac{\int_{-\infty}^{\infty} t h^2(t)dt}{\int_{-\infty}^{\infty} h^2(t)dt} \quad . \tag{5.5}$$

See Eq. (165) of Ref. 16 and the accompanying discussion.

Substituting Eq. (5.2) into Eq. (3.43), the second of our locally stationary response spectrum conditions becomes

$$\left|\frac{d}{dt} \ln \sigma_f^2(t)\right| << \frac{1}{|\tau_h(f)|} \quad , \tag{5.6}$$

where we have ignored the ratio $\Phi_Z(f)/\Phi_Z(f)$ in Eq. (3.43). *The requirement of Eq. (5.6) means, roughly, that the relative changes in the time-varying variance $\sigma_f^2(t)$ of the "fast turbulence component" $w_f(t)$ in Eq. (2.3) must be small in comparison with unity when measured over time intervals of the order of the group delay $\tau_h(f)$ of h(t).*

For unit-impulse response functions h(t) that are even functions of time – i.e., h(-t) = h(t) – it can be shown from Eq. (5.4) that $\tau_h(f) \equiv 0$. [Recall the choice of our time origin of h(t) as defined by Eqs. (3.23) and (3.24).] For such

impulse-response functions, Eq. (5.6) is always satisfied; hence, the third condition given by Eq. (3.46) is required. This third condition depends on the ratio $m_h^{(2)}(f)/m_h^{(0)}(f)$, which is an energy-spectrum-weighted frequency-decomposition of the second moment of $h^2(t)$ normalized to unit area. Since, by our choice of the time origin of $h(t)$, the time centroid of $h^2(t)$ is zero, it follows that $m_h^{(2)}(f)/m_h^{(0)}(f)$ is a frequency decomposition of the second *central* moment of $h^2(t)$ (normalized to unit area). See p. 286 of Ref. 16. It is shown on pp. 283 and 284 of Ref. 16 that the quantity $m_h^{(2)}(f)/m_h^{(0)}(f)$ can be expressed in terms of the magnitude and phase of $H(f)$ by

$$\frac{m_h^{(2)}(f)}{m_h^{(0)}(f)} = \frac{1}{4\pi^2}\left\{\left[\frac{d\theta_h(f)}{df}\right]^2 - \frac{1}{2}\,\frac{d^2 \ln|H(f)|}{df^2}\right\} . \tag{5.7}$$

Because $m_h^{(2)}(f)/m_h^{(0)}(f)$ is an energy-spectrum-weighted frequency-decomposition of the second central moment of $h^2(t)$ [with $h^2(t)$ normalized to unit area], *the requirement of Eq. (3.46) means, roughly that the relative changes in the time-varying variance $\sigma_f^2(t)$ of the "fast turbulence component" $w_f(t)$ in Eq. (2.3) must occur slowly when measured over time intervals comparable to the nominal duration of $h^2(t)$.* Since the nominal duration of $h^2(t)$ governs the "correlation interval" of the response process, when the requirements of Eqs. (3.41), (3.43), and (3.46) are satisfied, the fractional changes in $\sigma_f^2(t)$ should be small over intervals comparable to the "correlation interval" of the response process $\{y(t)\}$.

Locally Stationary Requirements Expressed in Terms of Mean-Square Response

The left-hand side of each of our locally stationary requirements of Eqs. (3.41), (3.43), and (3.46) depends only on the behavior of $\ln\sigma_f^2(t)$. Furthermore, when like terms are cancelled in the right-hand sides of these same equations, it is evident that the right-hand side of Eq. (3.41) depends only on the spectrum of the *turbulence* component $z(t)$, whereas the right-hand sides of Eqs. (3.43) and (3.46) depend only on properties $m_h^{(0)}(f)$, $m_h^{(1)}(f)$, and $m_h^{(2)}(f)$ of the aircraft impulse-response function $h(t)$.

By integrating the series expansion of Eq. (3.38) over $-\infty < f < \infty$, we obtain an expression for the conditional mean-square aircraft response $E\{y^2(t)|\sigma_f\}$. It is immediately evident that, after this integration is carried out, the three locally stationary requirements of Eqs. (3.41), (3.43), and (3.46) are replaced by

$$\left|\frac{d^2 \ln\sigma_f^2(t)}{dt^2}\right| << 32\pi^2 \frac{\int_{-\infty}^{\infty} \Phi_z(f)|H(f)|^2 df}{\left|\int_{-\infty}^{\infty} \Phi_z^{(2)}(f)|H(f)|^2 df\right|} , \tag{5.8}$$

$$\left|\frac{d}{dt} \ln\sigma_f^2(t)\right| << \frac{\int_{-\infty}^{\infty} \Phi_z(f)\ m_h^{(0)}(f)df}{\left|\int_{-\infty}^{\infty} \Phi_z(f)\ m_h^{(1)}(f)df\right|} , \tag{5.9}$$

and

$$\left|\frac{d^2}{dt^2} \ln\sigma_f^2(t) + \left[\frac{d}{dt} \ln\sigma_f^2(t)\right]^2\right| << 2 \frac{\int_{-\infty}^{\infty} \Phi_z(f)\ m_h^{(0)}(f)df}{\left|\int_{-\infty}^{\infty} \Phi_z(f)\ m_h^{(2)}(f)df\right|} . \tag{5.10}$$

The right-hand side of each of the above three conditions now depends both on the spectrum $\Phi_z(f)$ of the turbulence component $z(t)$ and on one or more of the quantities $m_h^{(0)}(f) = |H(f)|^2$, $m_h^{(1)}(f)$, and $m_h^{(2)}(f)$ that are determined from the (complex) aircraft frequency-response function $H(f)$. However, the right-hand sides of Eqs. (5.8), (5.9), and (5.10) have the advantage that, for a given turbulence component $z(t)$ and aircraft, each is defined by a single number only. These conditions are somewhat less restrictive than those defined by Eqs. (3.41), (3.43), and (3.46).

Expression of Requirements in Terms of Autocorrelation Function of $\ln\sigma_f^2(t)$

The left-hand side of the three requirements of Eqs. (3.41), (3.43), and (3.46) or Eqs. (5.8), (5.9), and (5.10) are expressed in terms of the function $\ln\sigma_f^2(t)$, which, until now, has been considered to be known or specified. However, in the turbulence model of Eq. (2.3), $\sigma_f(t)$ is assumed to be a sample function drawn from a stationary stochastic process. We shall now express the left-hand side of the above three locally stationary conditions in terms of the autocorrelation function of $\ln\sigma_f^2(t)$.

For brevity of notation, let us define

$$v(t) \triangleq \ln\sigma_f^2(t) \quad . \tag{5.11}$$

Denote the autocorrelation function of $v(t) = \ln\sigma_f^2(t)$ by $R_v(\tau)$; i.e.,

$$R_v(\tau) \triangleq E\{v(t)\ v(t+\tau)\} \quad . \tag{5.12}$$

Then, denoting by a prime the derivative of a function with respect to its argument, we have

$$\begin{aligned} R_v'(\tau) &= E\{v(t)\ v'(t+\tau)\} \\ &= E\{v(t-\tau)\ v'(t)\} \quad , \end{aligned} \tag{5.13}$$

since $\{v(t)\}$ is a stationary process. A second differentiation yields

$$\begin{aligned} R_v''(\tau) &= E\{-v'(t-\tau)\ v'(t)\} \\ &= -E\{v'(t)\ v'(t+\tau)\} \quad . \end{aligned} \tag{5.14}$$

Hence,

$$-R_v''(0) = E\{[v'(t)]^2\} \quad ; \tag{5.15}$$

i.e., the mean square value of $v'(t)$ is equal to $-R_v''(0)$. Furthermore, replacing $v(t)$ by $v'(t)$ in Eq. (5.12) yields, using the

result of Eq. (5.15),

$$R_v^{(4)}(0) = E\{[v''(t)]^2\} \quad , \tag{5.16}$$

where the superscript on the left-hand side represents the fourth derivative. The results of Eqs. (5.15) and (5.16) are, of course, well known – e.g., p. 21 of Ref. 17. Since $v(t) = \ln\sigma_I^2(2)$ is a stationary process, $v'(t)$ and $v''(t)$ are stationary processes with zero mean values. Hence, the standard deviation of $v'(t)$ is $[-R_v''(0)]^{1/2}$, and the standard deviation of $v''(t)$ is $[R_v^{(4)}(0)]^{1/2}$.

Suppose now, by our << signs in Eqs. (5.8), (5.9), and (5.10), we mean that the first neglected terms in Eqs. (3.38) should be a factor of about three less than the terms they are being compared with when the *stochastic* left-hand sides of the three conditions of Eqs. (5.8), (5.9), and (5.10) are statistically rather large, say at a "level" of three times their standard deviations. These two factors of three give us a composite factor of three times three, which we shall round off to ten. Consequently, the signs << in Eqs. (5.8) to (5.10) require a factor of about ten for their validity, when the left-hand sides are replaced by their standard deviations. For engineering applications, choice of a factor of ten as this minimum value for << is about right – i.e., not overly conservative. Hence, using Eqs. (5.16) and (5.15), we may express the requirements of Eqs. (5.8) and (5.9) as

$$[R_v^{(4)}(0)]^{1/2} \le 3.2\pi^2 \frac{\int_{-\infty}^{\infty} \Phi_z(f)|H(f)|^2 df}{\left|\int_{-\infty}^{\infty} \Phi_z^{(2)}(f)|H(f)|^2 df\right|} \tag{5.17}$$

and

$$[-R_v''(0)]^{1/2} \le \frac{1}{10} \frac{\int_{-\infty}^{\infty} \Phi_z(f)\ m_h^{(0)}(f) df}{\left|\int_{-\infty}^{\infty} \Phi_z(f)\ m_h^{(1)}(f) df\right|} \quad . \tag{5.18}$$

With the condition of Eq. (5.10), the situation is somewhat more complicated. The expected value of the left-hand side (without absolute value signs) is not zero in this case. Furthermore, we require the standard deviation of the sum of the two quantities on the left-hand side, and these two quantities are not statistically independent.

To provide an expression for the standard deviation of the left-hand side of Eq. (5.10) that is of practical use, it is necessary to assume that $v'(t) = d(\ln\sigma_r^2)/dt$ is a Gaussian process. This assumption would not normally be justified; however, for the kinds of inequalities required here, the assumption should not cause appreciable error. When this Gaussian assumption for $v'(t)$ is made, it is shown in Appendix E of this report that the required variance for the left-hand side of Eq. (5.10) is

$$\mathrm{Var}\{v''(t) + [v'(t)]^2\} = R_v^{(4)}(0) + 2[R_v''(0)]^2 \quad . \tag{5.19}$$

Furthermore, the mean value of the left-hand side of Eq. (5.10) (with absolute value signs removed) is

$$E\{v''(t) + [v'(t)]^2\} = E\{[v'(t)]^2\} = -R_v''(0) \quad , \tag{5.20}$$

according to Eqs. (5.11) and (5.15). Hence, using the three times the standard deviation criterion discussed earlier, and recognizing that this must be added to the mean value, we shall want for the left-hand side of Eq. (5.10)

$$\{2[R_v''(0)]^2 + R_v^{(4)}(0)\}^{1/2} - \frac{1}{3} R_v''(0)$$

$$= -\frac{1}{3} R_v''(0)\left(1 + 3\left\{2 + \frac{R_v^{(4)}(0)}{[R_v''(0)]^2}\right\}^{1/2}\right) \quad , \tag{5.21}$$

which is 1/3 times (mean plus three times standard deviation) and where we have recognized the fact that $R_v''(0)$ is negative. Using the "three times three equals ten" rule discussed earlier, the requirement of Eq. (5.10) can now be expressed as

$$-\frac{1}{3} R_v''(0)\left(1 + 3\left\{2 + \frac{R_v^{(4)}(0)}{[R_v''(0)]^2}\right\}^{1/2}\right) \le \frac{1}{5}\,\frac{\int_{-\infty}^{\infty} \Phi_z(f)\, m_h^{(0)}(f)df}{\left|\int_{-\infty}^{\infty} \Phi_z(f)\, m_h^{(2)}(f)df\right|} \quad , \tag{5.22}$$

which is our third requirement. *Equations (5.17), (5.18), and (5.22) are the conditions required for confident engineering use of the locally stationary approximation of Eq. (3.40) to compute the mean square aircraft response as was done, for example, in Eqs. (4.16), (4.17), and (4.18).*

Method for Evaluation of Autocorrelation Function of $\ell n \sigma_f^2(t)$

The left-hand sides of Eqs. (5.17), (5.18), and (5.22) depend only on the second and fourth derivatives of the autocorrelation function $R_v(\tau)$ of $v(t) = \ell n \sigma_f^2(t)$ evaluated at $\tau = 0$. Hence, to determine if these three conditions are satisfied, we must evaluate the autocorrelation function $R_v(\tau)$ from measured turbulence records. This problem will be treated now.

Let us return to the turbulence model of Eq. (2.3). Consider a high-pass filtered version $w_h(t)$ of $w(t)$, where the filter cutoff frequency is sufficiently high to remove, for practical purposes, the "slow process" $w_s(t)$. Hence, we may write this high-pass filtered version of $w(t)$ as

$$w_h(t) = A\sigma_f(t)\, z_h(t) \quad , \tag{5.23}$$

where $z_h(t)$ is a high-pass filtered version of the component $z(t)$ in Eq. (2.3) assumed normalized so that $E\{z_h^2\} = 1$, A is the constant required for this normalization, and where it has been assumed in writing Eq. (5.23) that fluctuations in $\sigma_f(t)$ occur slowly in comparison with fluctuations in $z(t)$. It is presumed that $w_h(t)$ would be generated from a turbulence recording by digital filtering. Thus, we shall assume that the record $w_h(t)$ is available.

Let us now square Eq. (5.23) and then take its natural logarithm. These operations yield

$$\ell n\, w_h^2(t) = \ell n\, A^2 + \ell n\, \sigma_f^2(t) + \ell n\, z_h^2(t) \quad . \tag{5.24}$$

By assumption in our turbulence model of Eq. (2.3), $\{\sigma_f(t)\}$ and $\{z(t)\}$ are independent processes; therefore, the same is true of $\{\ell n\, \sigma_f^2(t)\}$ and $\{\ell n\, z_h^2(t)\}$. Hence, we may immediately write the covariance of Eq. (5.24), taken at observation times t and $t + \tau$, as

$$\begin{aligned} \operatorname{cov}[\ell n\, w_h^2(t), \ell n\, w_h^2(t+\tau)] &= \operatorname{cov}[\ell n\, \sigma_f^2(t), \ell n\, \sigma_f^2(t+\tau)] \\ &\quad + \operatorname{cov}[\ell n\, z_h^2(t), \ell n\, z_h^2(t+\tau)] \quad , \end{aligned} \tag{5.25}$$

where we have used the fact that the covariance of the sum of two independent variables is equal to the sum of the covariances and where we note that the dependence on the constant A has disappeared. According to the material contained in the previous section, our interest is in the autocorrelation function $v(t) = \ln \sigma_f^2(t)$, which is related to the covariance function of $v(t) = \ln \sigma_f^2(t)$ by

$$R_v(\tau) = \mathrm{cov}[v(t), v(t+\tau)] + \{E[v]\}^2 \quad . \tag{5.26}$$

Hence, we wish to solve Eq. (5.25) for $\mathrm{cov}[\ln \sigma_f^2(t), \ln \sigma_f^2(t+\tau)]$; i.e.,

$$\mathrm{cov}[\ln \sigma_f^2(t), \ln \sigma_f^2(t+\tau)] = \mathrm{cov}[\ln w_h^2(t), \ln w_h^2(t+\tau)] - \mathrm{cov}[\ln z_h^2(t), \ln z_h^2(t+\tau)] \quad . \tag{5.27}$$

The first term on the right-hand side of Eq. (5.27) can be numerically computed directly from the high-pass filtered version $w_h(t)$ of the original sample $w(t)$. Thus, in order to compute $R_v(\tau)$, $v = \ln \sigma_f^2$, we need an expression for the second term on the right-hand side of Eq. (5.27). To obtain this expression, we first recall that, by assumption, $z(t)$ is a stationary Gaussian process; therefore, $z_h(t)$ also is a stationary Gaussian process. In Sec. 6 of this report, we shall describe a method for computing the power spectral density of $z(t)$ from the measured turbulence samples $w(t)$. Let us denote this power spectral density by $\Phi_z(f)$, and denote by $H_h(f)$ the high-pass filter complex frequency-response function. Then, when the gain of this filter is appropriately adjusted* so that $E\{z_h^2\} = 1$, the power spectral density $\Phi_{z_h}(f)$ of $z_h(t)$ can be computed from $\Phi_z(f)$ by

$$\Phi_{z_h}(f) = \Phi_z(f)\,|H_h(f)|^2 \quad . \tag{5.28}$$

Hence, from a measured turbulence record $w(t)$, we shall have at our disposal the means to compute the power spectral density $\Phi_{z_h}(f)$ of the component $z_h(t)$. Therefore, by using Eqs. (5.26),

*It will become evident later that this assumption need not be satisfied in practice.

(5.27), and (5.28), we shall have at our disposal the means to compute $R_v(\tau)$ if we can obtain $\mathrm{cov}[\ell n\, z_h^2(t),\ \ell n\, z_h^2(t+\tau)]$ from the power spectral density $\Phi_{z_h}(f)$ of $z_h(t)$.

Let us define

$$u(t) \triangleq \ell n\, z_h^2(t) \quad . \tag{5.29}$$

Then, the covariance of $u(t)$, $u(t+\tau)$ is related to the autocorrelation function $R_u(\tau)$ by

$$\mathrm{cov}[u(t),\ u(t+\tau)] = R_u(\tau) - \{E[u]\}^2 \quad . \tag{5.30}$$

Thus, if we can compute the autocorrelation function $R_u(\tau)$ of the function $u(t) = \ell n\, z_h^2(t)$, we have at our disposal the means to compute $R_v(\tau)$.

The problem, therefore, is to compute the autocorrelation function $R_u(\tau)$ of the natural logarithm of $z_h^2(t)$, given that we know the power spectrum $\Phi_{z_h}(f)$ of $\{z_h(t)\}$ and under the assumption that $\{z_h(t)\}$ is a stationary Gaussian process with zero mean and unit variance. This problem is solved in Appendix F, where it is determined that

$$R_u(\tau) = 2 \arcsin^2 \rho_{z_h}(\tau) + (C + \ell n\, 2)^2 \quad , \tag{5.31}$$

where C is Euler's constant – i.e.,

$$C = 0.577215665\ldots \tag{5.32}$$

– and where $\rho_{z_h}(\tau)$ is the correlation coefficient* of the process $z_h(t)$; i.e.,

$$\rho_{z_h}(\tau) = \frac{R_{z_h}(\tau)}{R_{z_h}(0)} \quad . \tag{5.33}$$

*The formulation of Eq. (5.33) negates the requirement that the high-pass filter gain be adjusted so that $E\{z_h^2\} = 1$.

From Eqs. (5.31) and (5.33), it immediately follows that

$$\mathrm{cov}[\ln z_h^2(t), \ln z_h^2(t+\tau)] = 2 \arcsin^2[R_{z_h}(\tau)/R_{z_h}(0)]. \quad (5.34)$$

Finally, we note that $R_{z_h}(\tau)$ may be computed from $\Phi_{z_h}(f)$ by

$$R_{z_h}(\tau) = \int_{-\infty}^{\infty} \Phi_{z_h}(f)\, e^{i2\pi f\tau}\, df$$

$$= \int_{-\infty}^{\infty} \Phi_z(f)\, |H_h(f)|^2\, e^{i2\pi f\tau}\, df \quad , \qquad (5.35)$$

according to Eq. (5.28). *Equations (5.26), (5.27), (5.34), and (5.35) collectively provide the means to compute the autocorrelation function $R_v(\tau)$ of $v(t) = \ln \sigma_f^2(t)$ from the high-pass filtered version $w_h(t)$ of the turbulence record $w(t)$ and the power spectral density $\Phi_z(f)$ of the turbulence component $z(t)$:*

$$R_v(\tau) = \{E[\ln \sigma_f^2(t)]\}^2 + \mathrm{cov}[\ln w_h^2(t), \ln w_h^2(t+\tau)]$$

$$- 2 \arcsin^2[R_{z_h}(\tau)/R_{z_h}(0)] \quad , \qquad (5.36)$$

where $R_{z_h}(\tau)$ is to be computed from $\Phi_z(f)$ and the high-pass filter frequency-response function by Eq. (5.35). Notice that, in computing the derivatives $R_v''(\tau)$ and $R_v^{(4)}(\tau)$, the constant term $\{E[\ln \sigma_f^2(t)]\}^2$ in Eq. (5.36) becomes irrelevant. Furthermore, insofar as computation of the derivatives of $R_v(\tau)$ is concerned, we may replace $\mathrm{cov}[\ln w_h^2(t), \ln w_h^2(t+\tau)]$ by the autocorrelation function of $\ln w_h^2(t)$ for this same reason. The computation of $\mathrm{cov}[\ln w_h^2(t), \ln w_h^2(t+\tau)]$ is to be carried out directly from the high-pass filtered version of the turbulence record $w(t)$.

METHODS FOR COMPUTATION OF TURBULENCE METRICS FOR PREDICTION OF AIRCRAFT RESPONSE STATISTICS

In addition to the turbulence metrics $R_{z_h}(\tau)$ and $\mathrm{Cov}[\ln w_h^2(t), \ln w_h^2(t+\tau)]$ used to compute the second and fourth derivatives of the autocorrelation function $R_v(\tau)$ of $v(t) = \ln \sigma_f^2(t)$, which are required for verification of the locally stationary aircraft response assumptions, the material described in Sec. 4 of this report delineates several other turbulence metrics necessary to predict the response exceedance rates and probability density functions of an arbitrary aircraft response variable.

To predict the mean rate $N_+(y)$ of exceedance crossings with positive slope of the aircraft response past the "level" y, one must compute the parameters $\sigma^2_{y_z}$, $\sigma^2_{\dot{y}_z}$, $\sigma^2_{y_s}$, $\sigma^2_{\dot{y}_s}$; therefore, as is evident from Eqs. (4.8) and (4.22), one must have available the power spectra $\Phi_z(f)$ and $\Phi_{w_s}(f)$ of the turbulence components $z(t)$ and $w_s(t)$ in the model of Eq. (2.3). See Eqs. (4.17), (4.18), (4.20), and (4.21). These same two spectra are also required to predict the aircraft response probability density function $p(y)$. Furthermore, computation of $N_+(y)$, using Eq. (4.3) or $p(y)$ using Eq. (4.42), requires the probability density function $p(\sigma_f^2)$ of the time-varying variance $\sigma_f^2(t)$ in Eq. (2.3). On the other hand, using Eqs. (4.28) and (4.46) to compute $N_+(y)$ and $p(y)$, respectively, requires the central moments $\mu^{(k)}_{\sigma_f^2}$ of σ_f^2 defined by Eq. (4.29). In addition, in deriving our expressions for $N_+(y)$ and $p(y)$, it was assumed that the turbulence component $w_s(t)$ in Eq. (2.3) was Gaussian. To check the consistency of this assumption, we require the probability density function $p(w_s)$ of $w_s(t)$. A simple check of the Gaussian character of $p(w_s)$ can be made using a Gram-Charlier series representation of $p(w_s)$, which requires the first few moments of $w_s(t)$. Finally, it can be shown that when the locally stationary response conditions of Eqs. (3.43) and (3.46) or Eqs. (5.18) and (5.22) are not satisfied, the power spectral density $\Phi_{\sigma_f^2}(f)$ of the component $\sigma_f^2(t)$ is required to predict response exceedance rates and probability density functions. In summary, to predict the aircraft response statistics described above, one requires methods for computation of the power spectral densities $\Phi_z(f)$, $\Phi_{w_s}(f)$, and $\Phi_{\sigma_f^2}(f)$ of the turbulence components $z(t)$, $w_s(t)$, and $\sigma_f^2(t)$, respectively; methods also are required for computation of the central moments and probability density functions $\mu^{(k)}_{\sigma_f^2}$ and $\mu^{(k)}_{w_s}$, and $p(\sigma_f^2)$ and $p(w_s)$ of the turbulence components $\sigma_f^2(t)$ and $w_s(t)$. [The component $z(t)$ in the model of Eq. (2.3) is, by assumption, Gaussian.]

In this section, we describe methods for computation of the above turbulence metrics from recorded turbulence velocities.

Estimation of Power Spectra of $z(t)$ and $w_s(t)$

Let us first obtain an expression for the autocorrelation function $R_w(\tau)$ using the model of Eq. (2.3). Multiplying $w(t)$ by $w(t+\tau)$ and taking the expected value of the resulting expression yields

$$E\{w(t)\ w(t+\tau)\} = E\{w_s(t)\ w_s(t+\tau)\} + E\{w_s(t)\ \sigma_f(t+\tau)\ z(t+\tau)\}$$

$$+ E\{w_s(t+\tau)\ \sigma_f(t)\ z(t)\} + E\{\sigma_f(t)\ \sigma_f(t+\tau)\ z(t)\ z(t+\tau)\}$$

$$= E\{w_s(t)\ w_s(t+\tau)\} + E\{\sigma_f(t)\ \sigma_f(t+\tau)\}\ E\{z(t)\ z(t+\tau)\}, \qquad (6.1)$$

or

$$R_w(\tau) = R_{w_s}(\tau) + R_{\sigma_f}(\tau)\ R_z(\tau) \quad , \qquad (6.2)$$

where, in going to the second equality in Eq. (6.1), we have used the assumption that $w_s(t)$, $\sigma_f(t)$, and $z(t)$ are statistically independent and that $z(t)$ has an expected value of zero.

To interpret Eq. (6.2), let us examine Fig. 4, which was first discussed in Sec. 2. First, we note that the nominal correlation interval associated with the process $\sigma_f(t)$ is much larger than that associated with $z(t)$. To see this, examine, for example, the clumps of turbulence between approximately 9 min 45 sec and 10 min 45 sec in Fig. 4. During this interval, the term $\sigma_f(t)$ in our model of Eq. (2.3) monotonically grows to a maximum from a small value and then decays monotonically back to a small value. The nominal correlation interval associated with $\sigma_f(t)$ is of the order of 1/2 min. On the other hand, the nominal correlation interval of the high-frequency process $z(t)$ is of the order of 1 sec. Thus, there is an order of magnitude or more separating the correlation scales of $\sigma_f(t)$ and $z(t)$. Since $z(t)$ has, by assumption, zero mean value, *this difference between correlation scales means that $R_{\sigma_f}(\tau)$ is very nearly equal to $R_{\sigma_f}(0)$ over the range of values of τ where $R_z(\tau)$ is not negligible.* Hence, to a first and quite good approximation, we may replace Eq. (6.2) by

$$R_w(\tau) \approx R_{w_s}(\tau) + R_{\sigma_f}(0)\, R_z(\tau)$$

$$= R_{w_s}(\tau) + \overline{\sigma_f^2}\, R_z(\tau) \quad , \tag{6.3}$$

where $\overline{\sigma_f^2} \equiv R_{\sigma_f}(0)$ is the expected value of the stationary process $\sigma_f^2(t)$. Let us now compare the nominal correlation intervals of the processes $w_s(t)$ and $z(t)$ by further examination of Fig. 4. It is evident that the correlation interval associated with the low-frequency (large-amplitude) process $w_s(t)$ is at least of the same order of magnitude as that of the process $\sigma_f(t)$. *Consequently, we may expect $R_{w_s}(\tau)$ to be very nearly constant and equal to $R_{w_s}(0)$ over the range of values of τ where $R_z(\tau)$ is not negligible.*

These observations are borne out by Fig. 5, which is the autocorrelation function of the vertical record shown in Fig. 4.* Equation (6.3) and the above discussion indicate that it should be possible to approximate the autocorrelation function shown in Fig. 5 by a linear combination of $R_{w_s}(\tau)$ and $R_z(\tau)$ and that there should be an order of magnitude or more difference between the nominal correlation intervals associated with these two additive components. This situation is precisely what Fig. 5 illustrates. In fact, if we conceive of representing $R_{w_s}(\tau)$ for $\tau \geq 0$ by a Maclaurin series,

$$R_{w_s}(\tau) = R_{w_s}(0) + R'_{w_s}(0+)\,|\tau| + \frac{R''_{w_s}(0+)}{2}\,\tau^2 \quad , \tag{6.4}$$

then it is evident from Fig. 5 that the linear approximation gives a good representation of $R_{w_s}(\tau)$ out to a lag of about 10,000 ft (3048 m), whereas the quadratic approximation in Eq. (6.4) would appear to give a good approximation over the entire lag interval shown in Fig. 5. For lag values in the range of 1000 to 10,000 ft (304.8 m to 3048 m), the deviations from the straight line shown in Fig. 5 may be attributed to statistical

*Figure 5 is the autocorrelation function expressed in terms of spatial lag $\xi = V\tau$ rather than temporal lag τ, where V is the speed of the measurement aircraft. This linear scale factor on the ordinate has no effect on the concepts being discussed.

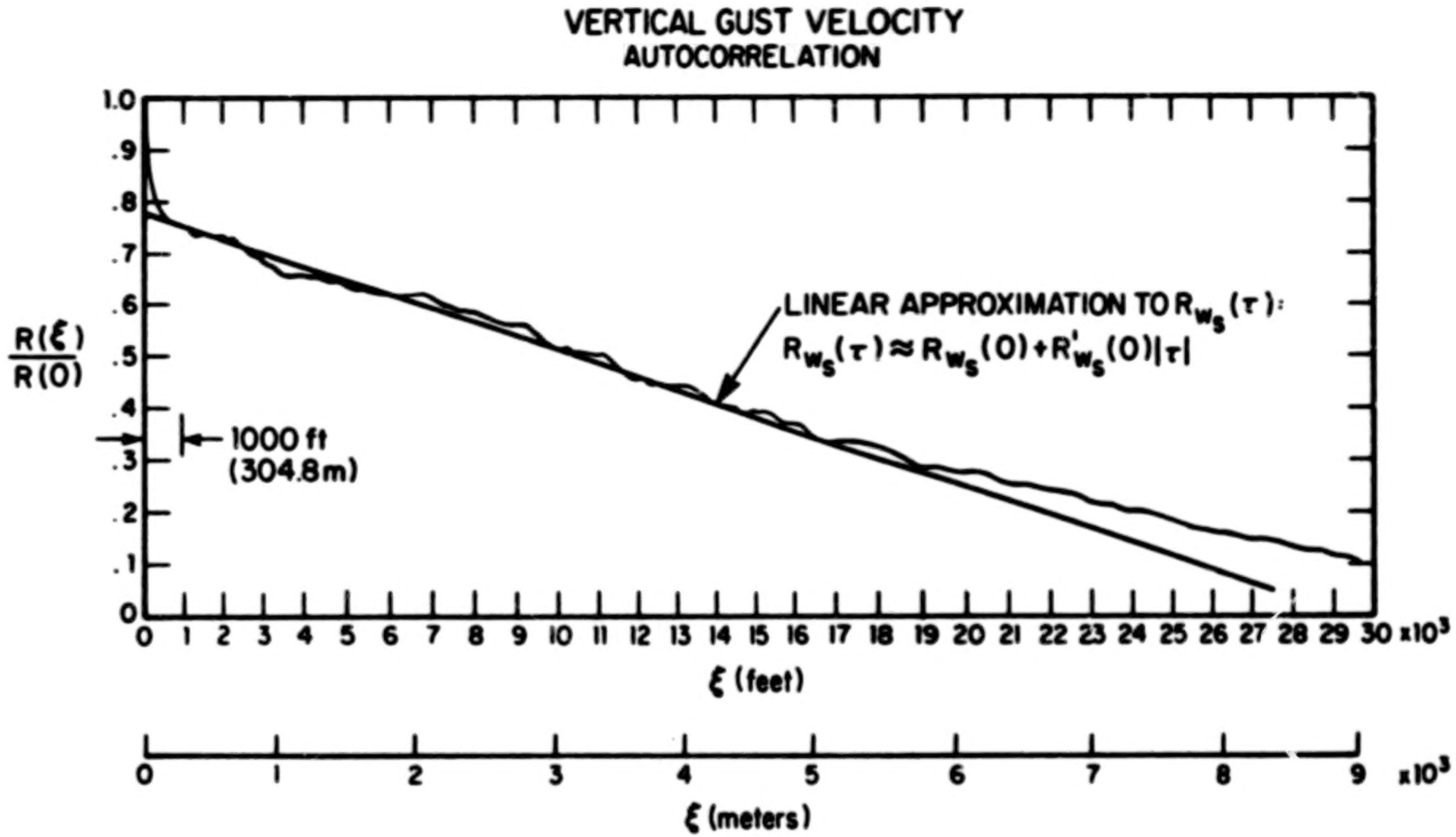

FIG. 5. AUTOCORRELATION FUNCTION OF VERTICAL RECORD SHOWN IN FIG. 4. [MOUNTAIN WAVE CONDITIONS. AIRCRAFT SPEED 197 m/sec (646 ft/sec).]

fluctuations in the *estimate* of $R_w(\tau)$ shown in Fig. 5 that are a result of the finite duration of the sample [which contains about 30 correlation intervals of the low-frequency component $w_s(t)$].

In the lag interval of Fig. 5 from 0 to 1000 ft (0 to 304.8 m), we see $\overline{\sigma_f^2}\, R_z(\tau)$ superimposed on $R_{w_s}(\tau)$, as is predicted by Eq. (6.3). Examination of Eq. (6.3) at $\tau = 0$ indicates that, by setting $R(0) = \sigma^2$ for both processes $w_s(\tau)$ and $z(\tau)$, we have

$$\sigma_w^2 = \sigma_{w_s}^2 + \overline{\sigma_f^2}\, \sigma_z^2$$

$$= \sigma_{w_s}^2 + \sigma_{w_f}^2 \quad , \tag{6.5}$$

where we have written

$$\sigma_{w_f}^2 = \overline{\sigma_f^2}\, \sigma_z^2 \quad , \tag{6.6}$$

which is consistent with the notation of Eq. (2.3). Examining the intersection of the straight-line approximation with the ordinate in Fig. 5, we see that

$$\frac{\sigma_{w_s}^2}{\sigma_w^2} \approx 0.78 \quad , \quad \frac{\sigma_{w_f}^2}{\sigma_w^2} \approx 0.22 \quad ; \tag{6.7a,b}$$

i.e., about 78% of the contribution to the mean square value of $w(t)$ in the vertical record is provided by the slow (low-frequency) component, whereas about 22% of the contribution is provided by the fast (high-frequency) component. These estimates are entirely consistent with the visual appearance of the vertical record in Fig. 4.

The above comments immediately suggest a first approximation to $R_z(\tau)$. Solving Eq. (6.3) for $R_z(\tau)$, we obtain

$$R_z(\tau) \approx \frac{1}{\overline{\sigma_f^2}} [R_w(\tau) - R_{w_s}(\tau)]$$

$$\approx \frac{1}{\overline{\sigma_f^2}} \{R_w(\tau) - [R_{w_s}(0) + R'_{w_s}(0)\, |\tau| + \ldots]\} \quad , \tag{6.8}$$

where, in the second line, we have substituted the Maclaurin series approximations to $R_{w_s}(\tau)$ given by Eq. (6.4) and where the coefficients $R_{w_s}(0)$, $R'_{w_s}(0)$, ... may be obtained either "by eye", as in the straight-line approximation of Fig. 5, or by a more sophisticated technique - e.g., least squares.

Once $R_z(\tau)$ is solved, using Eq. (6.8), we may find $\Phi_z(t)$ by forming the Fourier transform of $R_z(\tau)$. $\Phi_{w_s}(f)$ may then be found by subtracting $\overline{\sigma_f^2}\,\Phi_z(f)$ from the power spectrum $\Phi_w(f)$ of the original record, as the Fourier transform of Eq. (6.3) would indicate:

$$\Phi_{w_s}(f) \approx \Phi_w(f) - \overline{\sigma_f^2}\,\Phi_z(f) \ , \tag{6.9}$$

where $\overline{\sigma_f^2} = R_w(0) - R_{w_s}(0)$, since we have $R_z(0) \equiv 1$ by definition.

Two minor refinements of the above procedure will now be discussed. First, we note that, in the high-frequency region where the contribution of $\Phi_{w_s}(f)$ to $\Phi_w(f)$ is negligible, measured spectra consistently display a slope of -5/3, as predicted by the von Karman spectral forms. [The spectra $\Phi_{w_s}(f)$ contaminate $\Phi_z(f)$ in the low-frequency region.] This observation suggests that we *assume* that $\Phi_z(f)$ has the appropriate (transverse or longitudinal) von Karman form. Incorporation of the von Karman spectral form assumption for $\Phi_z(f)$ eliminates the problems associated with statistical fluctuations in estimation of the functional form of $\Phi_z(f)$. Estimation of $\Phi_z(f)$ is then reduced to estimation of the integral scale L of the component z(t), since $\sigma_z^2 = 1$ by definition.

A rough check of the von Karman form assumption for $\Phi_z(f)$ was performed as follows. Here, again, we have dealt with autocorrelation functions rather than spectra. The autocorrelation function dealt with is that computed from the vertical component of a sample case obtained from the NASA Langley MAT project, which is described in Ref. 18. This autocorrelation function is displayed in Fig. 6 on the same scale as that shown in Fig. 5. However, it is evident from Fig. 6 that the relative contribution of the low-frequency component $w_s(t)$ is much less in this case than it was in the case displayed in Fig. 5. The time history from which Fig. 6 was computed is displayed as the top trace in Fig. 7. Comparison of the top traces in Figs. 7 and 4 clearly shows that the relative contribution of $w_s(t)$ in the vertical trace in Fig. 7 is much less than that in the corresponding trace in Fig. 4.

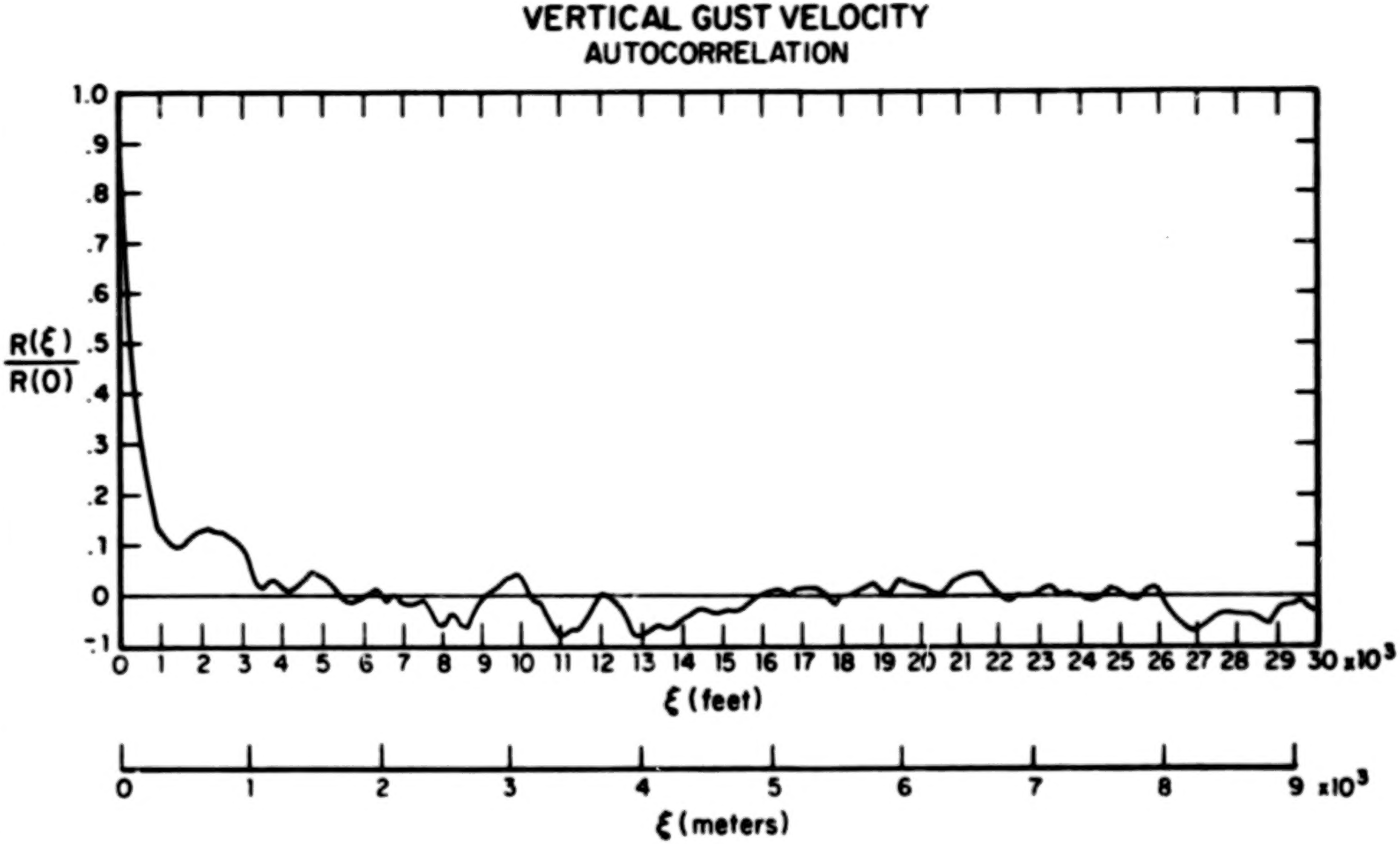

FIG. 6. AUTOCORRELATION FUNCTION OF VERTICAL RECORD SHOWN IN FIG. 7. [WIND SHEAR CONDITIONS. AIRCRAFT SPEED 188 m/sec (616 ft/sec).]

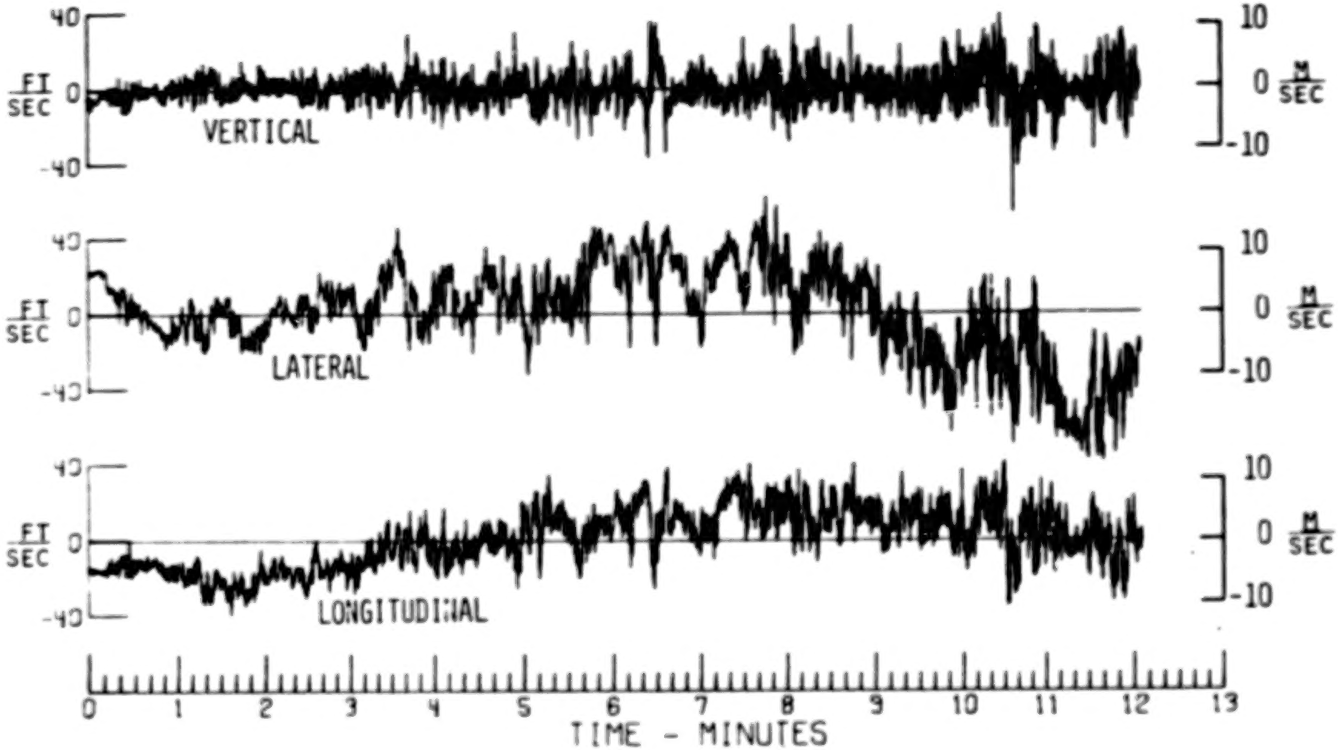

FIG. 7. TURBULENCE VELOCITY HISTORIES: WIND SHEAR CONDITIONS. [AIRCRAFT SPEED 188 m/sec (616 ft/sec).] (Ref. 2, Fig. 6, p. 283.)

The autocorrelation function shown in Fig. 6 is again displayed on an enlarged scale in Fig. 8. To perform a rough test of the von Karman assumption using this autocorrelation function, we estimated (crudely) that $(\sigma_{w_s}^2/\sigma_w^2) \approx 0.10$ for the autocorrelation function shown in Figs. 6 and 8. Furthermore, for this rough test, we retained only the constant term $R_{w_s}(0)$ in Eq. (6.4) as our approximation to $R_{w_s}(\tau)$. Using this approximation, Eq. (6.8) reduces our estimate of $R_z(\tau)$ to

$$R_z(\tau) \approx \frac{1}{\sigma_f^2} R_w(\tau) - \frac{1}{\sigma_f^2} R_{w_s}(0) \quad . \tag{6.10}$$

When normalized, this approximation to $R_z(\tau)$ is given by the continuous curve shown in Fig. 8 when considered as a function of the relabeled ordinate $R_z(\tau)/R_z(0)$.

The encircled points shown in Fig. 8 are points of the (transverse) von Karman autocorrelation function plotted as a function of $R_z(\tau)/R_z(0)$ for an integral scale of L = 170.7 m (560 ft). We refer the reader to p. 253 of Ref. 19 for the mathematical form of the von Karman transverse autocorrelation function, as well as for a graph of this function. The encircled points shown in Fig. 8 were obtained by careful reading of points off the von Karman transverse autocorrelation function plotted on p. 253 of Ref. 19, after the points had been scaled to the integral scale shown in Fig. 8. The fit of the circled points to the continuous curve is easily within our reading error of the original curve in Ref. 19. From Fig. 8, we must conclude that the von Karman curve gives an excellent fit to the (small τ) region of the curve that is not highly contaminated by the autocorrelation function $R_{w_s}(\tau)$ of the low-frequency component $w_s(t)$ in the model of Eq. (2.3).

A careful comparison of Figs. 5 and 6 indicates that no measurable difference in the components $R_z(\tau)$ in those two curves can be ascertained except for amplitude and integral-scale scale factors. Thus, it is likely that the assumption that $R_z(\tau)$ and $\Phi_z(f)$ have the appropriate von Karman forms is a good one.

One further comment about Fig. 8 is in order. The von Karman form falls almost exactly on the original curve where it first touches the *new* abscissa. Thus, one could estimate the integral scale of the component $z(t)$ quite accurately *in this case* by using the point where it first touches the *new* abscissa (i.e., the lowest circled point). However, the integral scale of the component $z(t)$ in the model of Eq. (2.3) would be considerably overestimated if one computed it from the first crossing of the *original* abscissa (as is commonly done).

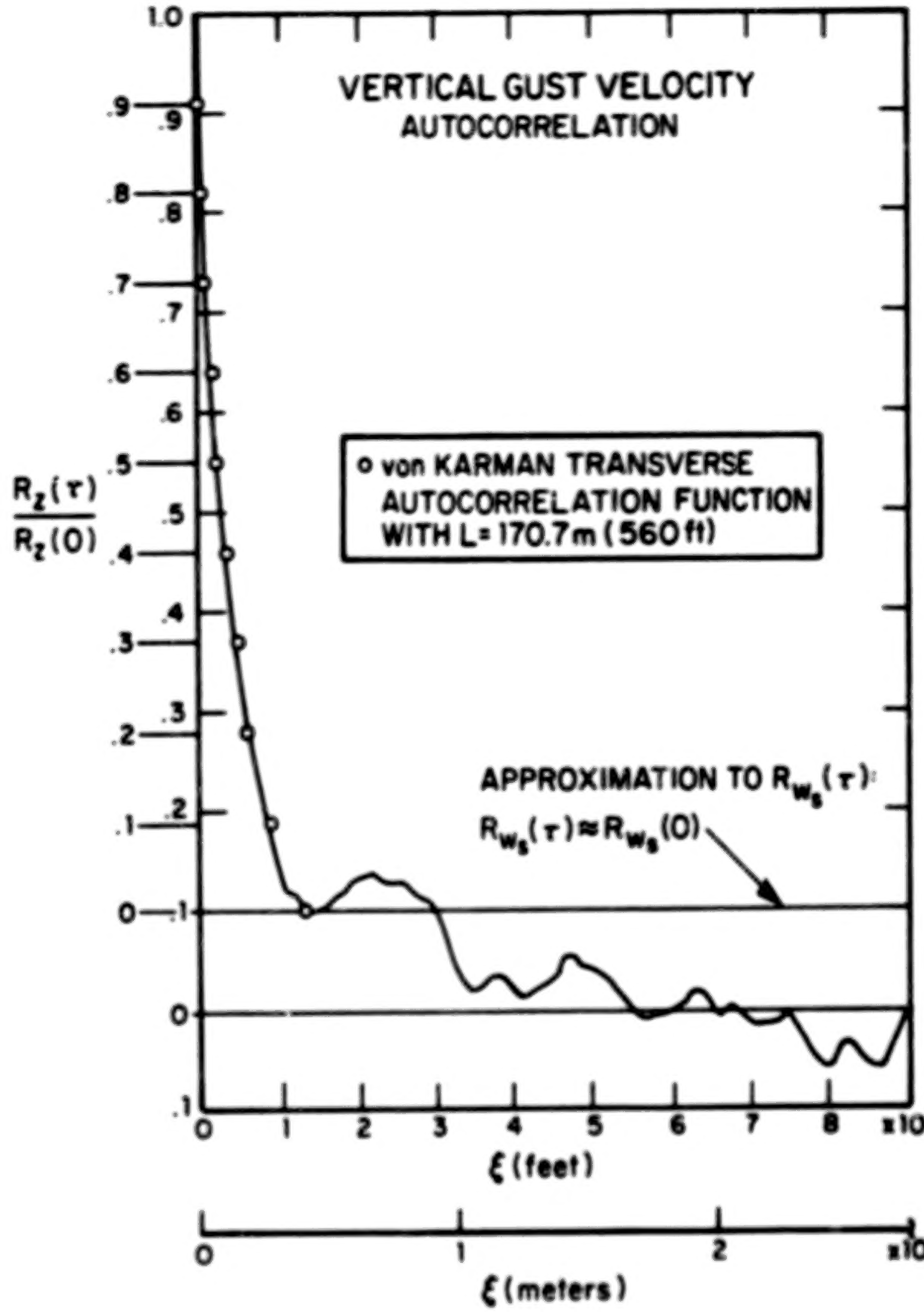

FIG. 8. AUTOCORRELATION FUNCTION OF FIG. 6 ON EXPANDED SCALES.

The second refinement to the method for estimation of $R_z(\tau)$ given by Eq. (6.8) has to do with the assumption that $R_{\sigma_f}(\tau) = R_{\sigma_f}(0) = \overline{\sigma_f^2}$ in the region of the τ axis where $R_z(\tau)$ is not negligible – i.e., the assumption that took us from Eq. (6.2) to Eq. (6.3). This assumption can be circumvented, because the normalized autocorrelation function

$$\rho_{\sigma_f}(\tau) \triangleq \frac{R_{\sigma_f}(\tau)}{R_{\sigma_f}(0)} \tag{6.11}$$

of $\sigma_f(t)$ can be estimated using the turbulence model of Eq. (2.3), as we shall show shortly. Before showing how $\rho_{\sigma_f}(\tau)$ can be computed, let us first summarize the above described procedure for estimating $\Phi_z(f)$ and $\Phi_{w_s}(f)$, where we shall assume that $\rho_{\sigma_f}(\tau)$ is known.

Procedure for estimating $\Phi_z(f)$ and $\Phi_{w_s}(f)$. Figure 9 illustrates the estimation procedure. The steps are: (1) Estimate a linear, quadratic, or possibly higher-order polynomial approximation to $R_{w_s}(\tau)$ in the neighborhood of $\tau = 0$, as is illustrated in Fig. 5. (2) Subtract this estimate of $R_{w_s}(\tau)$ from $R_w(\tau)$. The remaining function is our estimate of $R_{\sigma_f}(\tau)\ R_z(\tau)$, as is indicated by Eq. (6.2). Since, by definition, $R_z(0) \equiv 1$, we have

$$\overline{\sigma_f^2} = R_{\sigma_f}(0)\ R_z(0) \quad ; \tag{6.12}$$

i.e., we obtain an estimate of $\overline{\sigma_f^2}$ in this operation.* (3) Divide the estimate of $R_{\sigma_f}(\tau)\ R_z(\tau)$ by $\overline{\sigma_f^2}\ \rho_{\sigma_f}(\tau)$, which we shall describe how to estimate shortly. This division yields an estimate of $R_z(\tau)$. (4) Determine the integral scale that yields the best fit of the appropriate von Karman autocorrelation function to this estimate of $R_z(\tau)$. Note that $R_z(0)$ will be unity. (5) Subtract the product of this von Karman estimate of $R_z(\tau)$ and $\overline{\sigma_f^2}\ \rho_{\sigma_f}(\tau)$ – i.e., $R_{\sigma_f}(\tau)\ R_z(\tau)$ – from $R_w(\tau)$. This subtraction yields a new estimate of $R_{w_s}(\tau)$. (6) Examine the behavior of

*An independent alternative method for estimating $\overline{\sigma_f^2} = E(\sigma_f^2)$ is described in Appendix G.

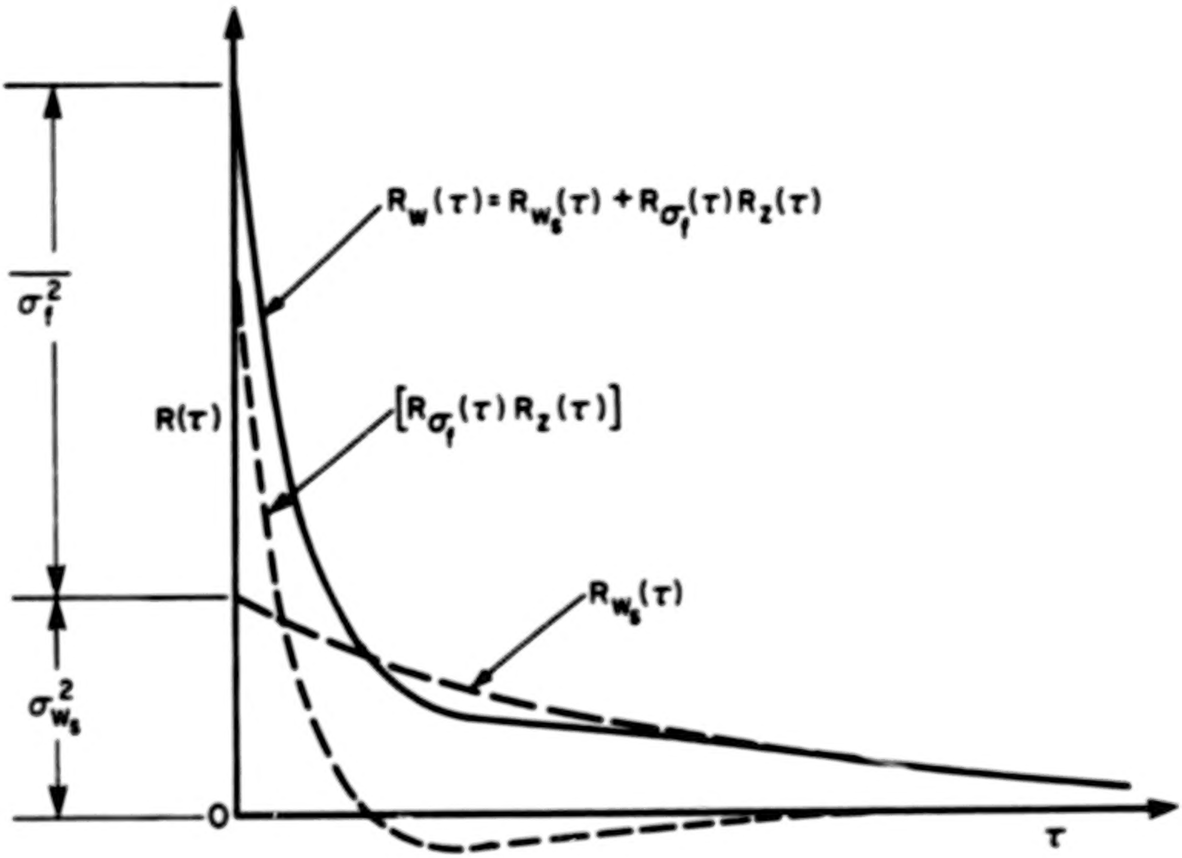

FIG. 9. IDEALIZED SKETCH OF AUTOCORRELATION FUNCTION OF ATMOSPHERIC TURBULENCE AND AUTOCORRELATION FUNCTION OF ITS COMPONENTS.

this new estimate of $R_{w_s}(\tau)$ in the vicinity of $\tau = 0$ to ascertain if there is any obvious contribution from $R_{\sigma_f}(\tau)\ R_z(\tau)$ remaining. If there is one, form a new estimate of $R_{w_s}(\tau)$ that eliminates the contribution of $R_{\sigma_f}(\tau)\ R_z(\tau)$ and repeat Steps (1) through (6). (7) The final estimate of $\Phi_z(f)$ is the appropriate von Karman spectral form with the integral scale provided by the above procedure. The final estimate of $\Phi_{w_s}(f)$ is the Fourier transform of the final estimate of $R_{w_s}(\tau)$ determined in Step (5).

Procedure for estimating $\rho_{\sigma_f}(\tau)$. Here, it will be assumed that there exists a frequency f_0 such that for all $f \geq f_0$ the contribution of the component $w_s(t)$ of the model of Eq. (2.3) to the spectrum of $w(t)$ is negligible in comparison with the contribution of the component $w_f(t) = \sigma_f(t)\ z(t)$. The frequency f_0 should be chosen on the low end of the frequency range of the portion of the spectrum $\Phi_w(f)$ that satisfies the -5/3 decay law of the von Karman spectrum. This value of f_0 would correspond to an inverse wavelength of about 3×10^{-3} cycles/ft (9.84×10^{-3} cycles/m) for the spectrum shown in Fig. 10, which is the wavenumber spectrum of the vertical record shown in Fig. 4, whose autocorrelation function is shown in Fig. 5.

The first step in estimating $\rho_{\sigma_f}(\tau)$ from a turbulence record $w(t)$ is to high-pass filter $w(t)$, where the high-pass filter attenuates all frequency components of $w(t)$ for $f < f_0$. Denote the filtered record by $w_h(t)$, as was done in Eq. (5.23):

$$w_h(t) = A'\ \sigma_f(t)\ z_h(t) \quad , \tag{6.13}$$

where, here, A' is an arbitrary positive constant and $z_h(t)$ is a high-pass filtered version of $z(t)$. Since $\sigma_f(t)$ is nonnegative, by definition, we may express the absolute value of $w_h(t)$ as

$$|w_h(t)| = A'\ \sigma_f(t)\ |z_h(t)| \quad . \tag{6.14}$$

Furthermore, since the record $w_h(t)$ is available for manipulation, we may compute its absolute value and find the autocorrelation function of $|w_h(t)|$, which, according to Eq. (6.14), is related to the autocorrelation functions of $\sigma_f(t)$ and $|z_h(t)|$ by

$$R_{|w_h|}(\tau) = (A')^2\ R_{\sigma_f}(\tau)\ R_{|z_h|}(\tau) \quad , \tag{6.15}$$

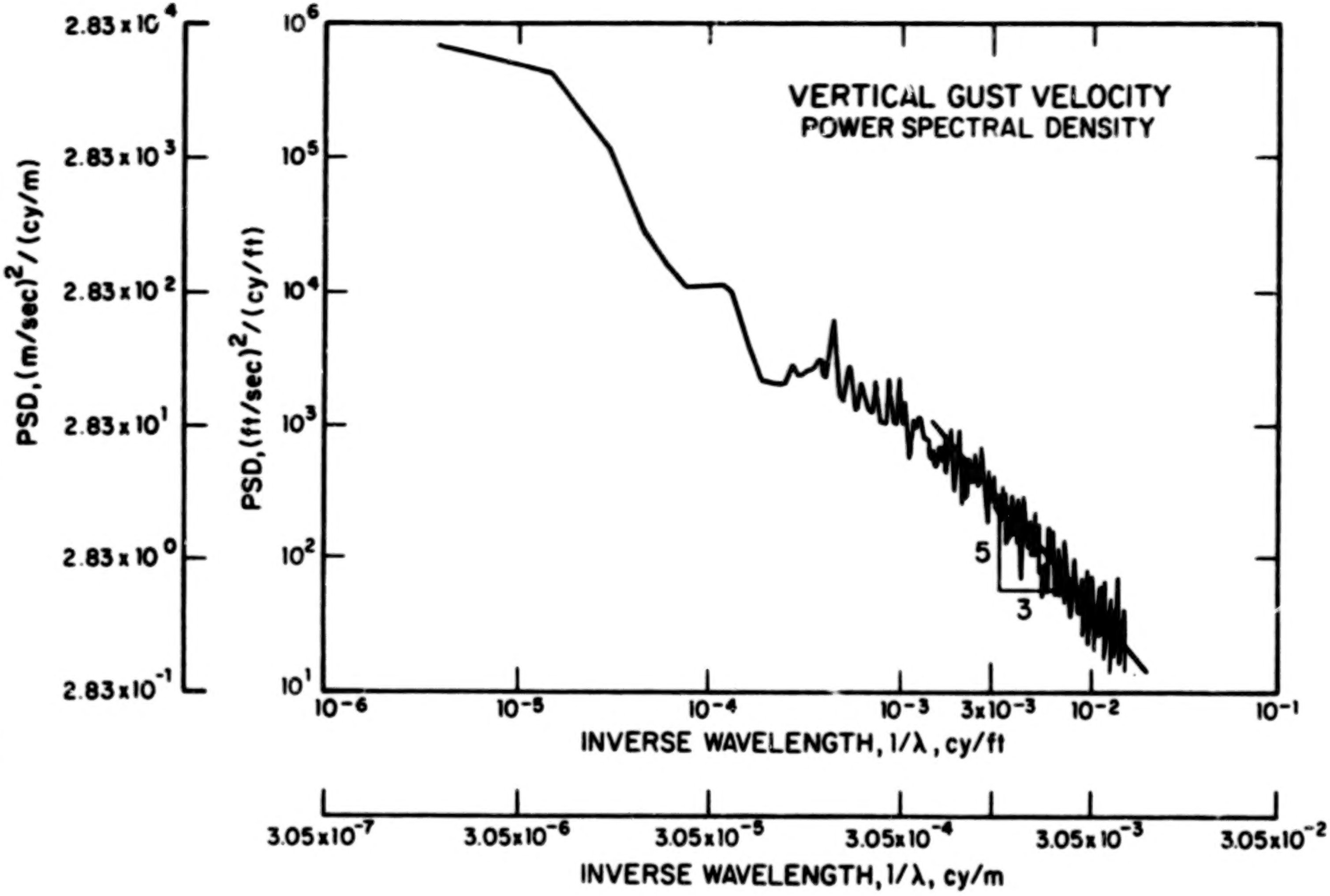

FIG. 10. POWER SPECTRUM OF VERTICAL RECORD SHOWN IN FIG. 4. (CORRESPONDING AUTO-CORRELATION FUNCTION IS SHOWN IN FIG. 5.) [MOUNTAIN WAVE CONDITIONS. AIRCRAFT SPEED 197 m/sec (646 ft/sec).]

where, in arriving at Eq. (6.15), the assumption that $\sigma_f(t)$ and $z_h(t)$ are statistically independent was used. Solving Eq. (6.15) for $R_{\sigma_f}(\tau)$ yields

$$R_{\sigma_f}(\tau) = \frac{R_{|w_h|}(\tau)}{(A')^2 \; R_{|z_h|}(\tau)} \quad ; \tag{6.16}$$

furthermore, dividing Eq. (6.16) by $R_{\sigma_f}(0)$ – and using the definition of Eq. (6.11) – yields

$$\rho_{\sigma_f}(\tau) = \frac{R_{|w_h|}(\tau)}{(A')^2 \; R_{\sigma_f}(0) \; R_{|z_h|}(\tau)} \quad . \tag{6.17}$$

It was shown in Ref. 5 that only rarely can we not assume that the autocorrelation function of $w_h(t)$ in Eq. (6.13) is well represented by

$$R_{w_h}(\tau) \simeq (A')^2 \; E\{\sigma_f^2\} \; R_{z_h}(\tau)$$

$$= (A')^2 \; R_{\sigma_f}(0) \; R_{z_h}(\tau) \quad . \tag{6.18}$$

Equations (6.17) and (6.18) provide us with the means to estimate the normalized autocorrelation function $\rho_{\sigma_f}(\tau)$. The numerator of Eq. (6.17) can be computed directly from the absolute value of the filtered turbulence record. Also, we can compute $R_{w_h}(\tau)$ from the same (unrectified) record. Furthermore, since $z(t)$, by assumption, is a stationary Gaussian process, $z_h(t)$ also is stationary and Gaussian. We therefore can compute the quantity $(A')^2 \; R_{\sigma_f}(0) \; R_{|z_h|}(\tau)$ from the measured autocorrelation function $R_{w_h}(\tau)$ by using the well known relationship [e.g., p. 166 of Ref. 8]

$$(A')^2 \; R_{\sigma_f}(0) \; R_{|z_h|}(\tau) = \frac{2}{\pi}\Bigg\{\sqrt{[R_{w_h}(0)]^2 - [R_{w_h}(\tau)]^2}$$

$$+ R_{w_h}(\tau) \arccos\left[-R_{w_h}(\tau)/R_{w_h}(0)\right] - \frac{\pi}{2} R_{w_h}(\tau)\Bigg\} \quad . \tag{6.19}$$

*Thus, according to Eq. (6.17), $\rho_{\sigma_f}(\tau)$ may be estimated by dividing the autocorrelation function $R_{|w_h|}(\tau)$ [which is to be computed directly from the absolute value $|w_h(t)|$ of the filtered record $w_h(t)$] by the right-hand side of Eq. (6.19) [which is computed from $R_{w_h}(\tau)$]. The autocorrelation function $R_{w_h}(\tau)$ is to be computed directly from the (unrectified) record $w_h(t)$.**

We may check the consistency of the above method by evaluating Eq. (6.17) at $\tau = 0$ and $\tau = \infty$. First, consider $\tau = 0$. Since $R_{|w_h|}(0) = E\{|w_h|^2\} = E\{w_h^2\}$, Eq. (6.17) yields, at $\tau = 0$,

$$\rho_{\sigma_f}(0) = \frac{E\{w_h^2\}}{(A')^2\, R_{\sigma_f}(0)\, R_{|z_h|}(0)} \quad . \tag{6.20}$$

*The approximation made in Eq. (6.18) can be avoided if the "arcsin law" is employed to compute $R_{|z_h|}(\tau)$ in the denominator of Eq. (6.17). We used the arcsin law successfully with turbulence data in work reported in Ref. 6 (e.g., pp. 16-21). In using the arcsin law in the computation of $\rho_{\sigma_f}(\tau)$, one must replace the denominator of the right-hand side of Eq. (6.17) by

$$\frac{2}{\pi} R_{w_h}(0) \left\{ \sqrt{1-[R_{z_h'}(\tau)]^2} + R_{z_h'}(\tau) \arccos[-R_{z_h'}(\tau)] - \frac{\pi}{2} R_{z_h'}(\tau) \right\} ,$$

where

$$R_{z_h'}(\tau) \triangleq \sin[\frac{\pi}{2} R_0(\tau)] \quad ,$$

where $R_0(\tau)$ is the autocorrelation function of the hard-clipped version of $w_h(\tau)$ that is formed by setting the hard-clipped waveform equal to +1 where it is positive and -1 where it is negative.

The denominator of Eq. (6.20) must be evaluated from Eq. (6.19); for $\tau = 0$, using the fact that $\arccos(-1) = \pi$, we find from Eq. (6.19) that

$$(A')^2 R_{\sigma_f}(0) R_{|z_h|}(0) = \frac{2}{\pi}\left\{ \pi R_{w_h}(0) - \frac{\pi}{2} R_{w_h}(0)\right\}$$

$$= R_{w_h}(0) = E\{w_h^2\} \quad ; \tag{6.21}$$

hence, by combining Eqs. (6.20) and (6.21), we have $\rho_{\sigma_f}(0) \equiv 1$, which is, of course, correct.

Now let us evaluate Eq. (6.17) at $\tau = \infty$. Recognizing that $R_{|w_h|}(\infty) = \{E[|w_h|]\}^2$, we see that Eq. (6.17) yields, at $\tau = \infty$,

$$\rho_{\sigma_f}(\infty) = \frac{\{E[|w_h|]\}^2}{(A')^2 R_{\sigma_f}(0) R_{|z_h|}(\infty)} \quad . \tag{6.22}$$

Evaluating the denominator using Eq. (6.19) and recognizing that $R_{w_h}(\infty) = 0$, we have

$$(A')^2 R_{\sigma_f}(0) R_{|z_h|}(\infty) = \frac{2}{\pi} \{R_{w_h}(0)\} = \frac{2}{\pi} E[w_h^2] \quad . \tag{6.23}$$

Combining Eqs. (6.22) and (6.23) yields

$$\rho_{\sigma_f}(\infty) = \frac{\{E[|w_h|]\}^2}{\frac{2}{\pi} E[w_h^2]} \quad . \tag{6.24}$$

However, from Eq. (6.14), we have

$$E[|w_h|] = A' E[\sigma_f] E[|z_h|] \quad , \tag{6.25}$$

whereas from Eq. (6.13) we have

$$E[w_h^2] = (A')^2 E[\sigma_f^2] E[z_h^2] \quad . \tag{6.26}$$

Substituting Eqs. (6.25) and (6.26) into Eq. (6.24) yields

$$\rho_{\sigma_f}(\infty) = \frac{\{E[\sigma_f]\}^2}{E[\sigma_f^2]} \frac{\{E[|z_h|]\}^2}{\frac{2}{\pi} E[z_h^2]} \quad . \tag{6.27}$$

However, $z_h(t)$ is a Gaussian process with zero mean value; hence, we have [e.g., Eq. (4.4-25) on p. 166 of Ref. 8]

$$E[|z_h|] = \sqrt{\frac{2}{\pi} E[z_h^2]} \quad . \tag{6.28}$$

Therefore, Eqs. (6.27) and (6.28) yield

$$\rho_{\sigma_f}(\infty) = \frac{\{E[\sigma_f]\}^2}{E[\sigma_f^2]} \quad , \tag{6.29}$$

which is the correct value. The method of Eqs. (6.17) and (6.19) therefore checks, exactly, in the limiting cases $\tau = 0$ and $\tau = \infty$.

Estimation of Power Spectrum of $\sigma_f^2(t)$

A procedure similar to that described above can be used to estimate the autocorrelation function $R_{\sigma_f^2}(\tau)$ of $\sigma_f^2(t)$. Squaring Eq. (6.13), we have

$$w_h^2(t) = (A')^2 \ \sigma_f^2(t) \ z_h^2(t) \quad . \tag{6.30}$$

From the assumed statistical independence of $\sigma_f^2(t)$ and $z_h^2(t)$, it follows that

$$R_{w_h^2}(\tau) = (A')^4 \ R_{\sigma_f^2}(\tau) \ R_{z_h^2}(\tau) \quad , \tag{6.31}$$

where we have defined

$$R_{w_h^2}(\tau) = E\{w_h^2(t) \ w_h^2(t+\tau)\} \quad , \tag{6.32}$$

with comparable definitions holding for $R_{\sigma_f^2}(\tau)$ and $R_{z_h^2}(\tau)$. Solving Eq. (6.31) for $R_{\sigma_f^2}(\tau)$ yields

$$R_{\sigma_f^2}(\tau) = \frac{R_{w_h^2}(\tau)}{(A')^4 \; R_{z_h^2}(\tau)} \quad . \tag{6.33}$$

By squaring a given high-pass filtered recording $w_h(t)$, we can compute $R_{w_h^2}(\tau)$ directly. To compute $R_{z_h^2}(\tau)$, we begin with the approximation of Eq. (6.18), which can be expressed as

$$R_{w_h}(\tau) \approx (A')^2 \; E[\sigma_f^2] \; R_{z_h}(\tau) \quad , \tag{6.34}$$

which is equivalent to the expression

$$E\{w_h(t) \; w_h(t+\tau)\} \approx E\{[A'\sqrt{E(\sigma_f^2)} \; z_h(t)] \; [A'\sqrt{E(\sigma_f^2)} \; z_h(t+\tau)]\} \; . \tag{6.35}$$

Now, *if Eq. (6.35) were an exact expression,* it would follow that

$$R_{w_h^2}(\tau) = (A')^4 \; \{E[\sigma_f^2]\}^2 \; R_{z_h^2}(\tau) \quad , \tag{6.36}$$

from which it would follow that

$$R_{z_h^2}(\tau) = \frac{R_{w_h^2}(\tau)}{(A')^4 \; \{E[\sigma_f^2]\}^2} \quad . \tag{6.37}$$

But, from Eq. (4.4-26) on p. 166 of Ref. 8, we have for stationary Gaussian processes $\{w_h(t)\}$,

$$R_{w_h^2}(\tau) = [R_{w_h}(0)]^2 + 2[R_{w_h}(\tau)]^2 \quad . \tag{6.38}$$

Substitution of Eq. (6.38) into Eq. (6.37) yields

$$R_{z_h^2}(\tau) \approx \frac{[R_{w_h}(0)]^2 + 2[R_{w_h}(\tau)]^2}{(A')^4 \{E[\sigma_f^2]\}^2} , \tag{6.39}$$

the validity of which depends on the stationary Gaussian assumption for $\{z_h(t)\}$ and the approximation of Eq. (6.34). Combining Eqs. (6.33) and (6.39) yields, finally,

$$R_{\sigma_f^2}(\tau) \approx \{E[\sigma_f^2]\}^2 \frac{R_{w_h^2}(\tau)}{[R_{w_h}(0)]^2 + 2[R_{w_h}(\tau)]^2} , \tag{6.40}$$

which is the desired expression.* Notice that $R_{w_h}(\tau)$ can be

*The approximation of Eq. (6.34) can be eliminated in deriving an expression for $R_{\sigma_f^2}(\tau)$ using the "arcsin law" as was the case with $\rho_{\sigma_f}(\tau)$ considered in the previous footnote. From Eq. (6.33), we have, exactly,

$$\rho_{\sigma_f^2}(\tau) \triangleq \frac{R_{\sigma_f^2}(\tau)}{R_{\sigma_f^2}(0)} = \frac{R_{w_h^2}(\tau)}{R_{w_h^2}(0)} \frac{R_{z_h^2}(0)}{R_{z_h^2}(\tau)} .$$

Applying Eq. (6.38) to the Gaussian process $z_h(t)$ yields

$$R_{z_h^2}(\tau) = [R_{z_h}(0)]^2 \{1 + 2[\rho_{z_h}(\tau)]^2\} .$$

Using the "arcsin law", we have

$$\rho_{z_h}(\tau) = \sin[\tfrac{\pi}{2} R_0(\tau)] ,$$

where $R_0(\tau)$ was defined in the last footnote. The above expressions can be evaluated to yield $\rho_{\sigma_f^2}(\tau)$ and, therefore, $R_{\sigma_f^2}(\tau)$ in terms of $R_{\sigma_f^2}(0)$. To compute $R_{\sigma_f^2}(0)$, we note that

$$\rho_{\sigma_f^2}(\infty) = \frac{R_{\sigma_f^2}(\infty)}{R_{\sigma_f^2}(0)} = \frac{\{E[\sigma_f^2]\}^2}{R_{\sigma_f^2}(0)} ,$$

which can be solved for $R_{\sigma_f^2}(0)$ in terms of $\rho_{\sigma_f^2}(\infty)$ and $\{E[\sigma_f^2]\}^2$.

evaluated directly from the high-pass filtered turbulence sample $w_h(t)$, whereas $R_{w_h^2}(\tau)$ can be evaluated directly from the squared sample $w_h^2(t)$. $E\{\sigma_f^2\}$ can be evaluated by the method illustrated in Fig. 9 or by the method described in Appendix G. Notice also that, from the definition of $R_{\sigma_f^2}(\tau)$, we must have

$$\lim_{\tau\to\infty} R_{\sigma_f^2}(\tau) = \{E[\sigma_f^2]\}^2 \quad . \tag{6.41}$$

To check the consistency of Eq. (6.40), we shall evaluate it for the limiting cases $\tau = 0$ and $\tau = \infty$. For $\tau = 0$, Eq. (6.40) yields

$$R_{\sigma_f^2}(0) = \{E[\sigma_f^2]\}^2 \frac{E\{w_h^4\}}{\{E[w_h^2]\}^2 + 2\{E[w_h^2]\}^2}$$

$$= \{E[\sigma_f^2]\}^2 \frac{E\{w_h^4\}}{3\{E[w_h^2]\}^2} \quad . \tag{6.42}$$

However, from Eq. (6.13), using the statistical independence of $\sigma_f(t)$ and $z_h(t)$, we have

$$E\{w_h^4\} = (A')^4 \, E[\sigma_f^4] \, E[z_h^4] \tag{6.43}$$

and

$$E[w_h^2] = (A')^2 \, E[\sigma_f^2] \, E[z_h^2] \quad . \tag{6.44}$$

Substituting these quantities into Eq. (6.42) and cancelling identical quantities in the numerator and denominator yield

$$R_{\sigma_f^2}(0) = E[\sigma_f^4] \frac{E[z_h^4]}{3\{E[z_h^2]\}^2} \quad . \tag{6.45}$$

However, $z_h(t)$ is assumed to be a stationary Gaussian process with zero mean value. It follows (e.g., p. 221 of Ref. 10) that

$$E[z_h^4] = 3\{[z_h^2]\}^2 \quad ; \tag{6.46}$$

hence, we have from Eqs. (6.45) and (6.46)

$$R_{\sigma_f^2}(0) = E[\sigma_f^4] \quad , \tag{6.47}$$

which, of course, is an identity. Thus, Eq. (6.40) yields the correct value at $\tau = 0$.

For $\tau = \infty$, it follows directly from Eq. (6.40) that we may write

$$\begin{aligned} R_{\sigma_f^2}(\infty) &= \{E[\sigma_f^2]\}^2 \frac{\{E[w_h^2]\}^2}{\{E[w_h^2]\}^2} \\ &= \{E[\sigma_f^2]\}^2 \quad , \end{aligned} \tag{6.48}$$

which is the correct limiting value given by Eq. (6.41). Consequently, Eq. (6.40) provides the correct exact limiting values at $\tau = 0$ and $\tau = \infty$.

To obtain the power spectral density $\Phi_{\sigma_f^2}(f)$ *of* $\sigma_f^2(t)$, *we Fourier transform Eq. (6.40). Recognizing Eq. (6.41), we may express* $\Phi_{\sigma_f^2}(f)$ as

$$\Phi_{\sigma_f^2}(f) = \{E[\sigma_f^2]\}^2 \left\{ \delta(f) + \int_{-\infty}^{\infty} \left(\frac{R_{w_h^2}(\tau) - [R_{w_h}(0)]^2 - 2[R_{w_h}(\tau)]^2}{[R_{w_h}(0)]^2 + 2[R_{w_h}(\tau)]^2} \right) e^{-i2\pi f\tau} d\tau \right\}, \tag{6.49}$$

where $\delta(f)$ *is the Dirac delta function and where* $\{E[\sigma_f^2]\}^2$ *may be evaluated using the method described in Appendix G or from the method illustrated in Fig. 9. All that are required to evaluate the integrand in Eq. (6.49) are the autocorrelation functions of* $w_h(t)$ *and* $w_h^2(t)$, *both of which can be evaluated directly from the high-pass filtered turbulence sample* $w_h(t)$. *The Fourier transform in Eq. (6.49) would, of course, be carried out numerically.*

Estimation of Moments and Probability Density of $\sigma_f^2(t)$

Moments of σ_f^2. Let us now turn to the development of methods for determination of the moments and probability density function of $\sigma_f^2(t)$. The moments $\mu_{\sigma_f^2}^{(n)}$ required for computation of the response exceedance rates $N_+(y)$ and probability density function $p(y)$ in Eqs. (4.28) and (4.46) are the *central* moments — i.e., moments taken about the mean of σ_f^2. See Eq. (4.29). Here, we shall develop a method for determining the moments $M_{\sigma_f^2}^{(k)}$ taken about the origin — i.e.,

$$M_{\sigma_f^2}^{(k)} \triangleq \int_0^\infty (\sigma_f^2)^k \; p(\sigma_f^2) d\sigma_f^2 \tag{6.50a}$$

$$\equiv E\{[\sigma_f^2]^k\} \quad . \tag{6.50b}$$

It is easy to show [e.g., Eqs. (98) and (99) on pp. 273 and 274 of Ref. 16] that the *nth* central moment can be computed from the moments $M_{\sigma_f^2}^{(k)}$, $k = 0,1,2,\ldots,n$ by the formula

$$\mu_{\sigma_f^2}^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \left(-M_{\sigma_f^2}^{(1)}\right)^{n-k} M_{\sigma_f^2}^{(k)} \quad , \; n = 1,2,3,\ldots \; , \tag{6.51}$$

where we note that $M_{\sigma_f^2}^{(0)}$ is the area under the probability density function; hence,

$$M_{\sigma_f^2}^{(0)} \equiv 1 \quad , \tag{6.52}$$

and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{6.53}$$

are the binomial coefficients. Notice, in particular, that for $n = 1$ and 2, Eq. (6.51) yields

$$\mu_{\sigma_f^2}^{(1)} = 0 \quad , \; \mu_{\sigma_f^2}^{(2)} = M_{\sigma_f^2}^{(2)} - \left(M_{\sigma_f^2}^{(1)}\right)^2 \quad . \tag{6.54a,b}$$

Equation (6.54a) states that the first central moment is zero, whereas Eq. (6.54b) is the familiar expression for the second central moment, which is the variance. [See Eq. (4.23).]

Let us turn now to developing a method for determining the moments $M^{(n)}_{\sigma_f^2}$, n = 1,2,3,... defined by Eq. (6.50). To do so, we shall again assume that we have available the high-pass filtered turbulence sample $w_h(t)$ described earlier, which has the form of Eq. (6.13), where A' is an arbitrary positive constant and $z_h(t)$ is a high-pass filtered version of the Gaussian turbulence component z(t) defined by Eq. (2.3). Let us now write an expression for the *nth* moment of $w_h^2(t)$, which is expressed in terms of $\sigma_f^2(t)$ and $z_h^2(t)$ by Eq. (6.30). Using the fact that $\sigma_f(t)$ and z(t) are assumed to be statistically independent, it follows immediately that we can write the *nth* moment of Eq. (6.30) as

$$E\{[w_h^2]^n\} = (A')^{2n}\, E\{[\sigma_f^2]^n\}\, E\{[z_h^2]^n\} \quad , \tag{6.55}$$

from which we may solve for $E\{[\sigma_f^2]^n\}$ as

$$E\{[\sigma_f^2]^n\} = \frac{E\{[w_h^2]^n\}}{(A')^{2n}\, E\{[z_h^2]^n\}} \quad . \tag{6.56}$$

Since we have assumed that the high-pass filtered sample $w_h(t)$ is available, we can numerically square it and then compute the first several moments $E\{[w_h^2]^n\}$, n = 1,2,3,.... Furthermore, since z(t) is assumed to be Gaussian, $z_h(t)$ also is Gaussian; hence, we can compute the moments $E\{[z_h^2]^n\}$ in terms of $E\{z_h^2\}$, as we shall now show.

It is known (e.g., pp. 233-236 of Ref. 9) that the probability density function of the square of the Gaussian variate z_h^2 is given by the chi-square density with one degree-of-freedom. Letting

$$x = z_h^2 \tag{6.57}$$

and

$$\sigma^2 = E\{z_h^2\} \quad , \tag{6.58}$$

we have, from p. 236 of Ref. 9, for the probability density of z_h^2

$$p_{z_h^2}(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\ \sigma}\ x^{-\frac{1}{2}}\ e^{-x/(2\sigma^2)} & , x > 0 \\ 0 & , x < 0 \quad . \end{cases} \tag{6.59}$$

To determine an expression for the moments of z_h^2,

$$M_{z_h^2}^{(n)} \equiv E\{[z_h^2]^n\} = M_x^{(n)} \quad , \tag{6.60}$$

we shall want to consider the normalized variable, which is defined from Eqs. (6.57) and (6.58) as

$$\xi \triangleq \frac{z_h^2}{\sigma^2} = \frac{x}{\sigma^2} \quad . \tag{6.61}$$

Let $p_\xi(\xi)$ denote the probability density of ξ. Then, since

$$p_{z_h^2}(x)dx = p_\xi(\xi)d\xi \quad , \tag{6.62}$$

it follows immediately that the moments $M_x^{(n)}$ of z_h^2 can be computed from the moments $M_\xi^{(n)}$ of ξ by

$$M_x^{(n)} \triangleq \int_0^\infty x^n\ p_{z_h^2}(x)dx = \int_0^\infty (\sigma^2\xi)^n\ p_\xi(\xi)d\xi \tag{6.63a}$$

$$= \sigma^{2n}\ M_\xi^{(n)} \quad , \tag{6.63b}$$

where we have used Eqs. (6.61) and (6.62) and where $M_\xi^{(n)}$ denotes the nth moment of the variable $\xi = x/\sigma^2$.

Using Eqs. (6.61) and (6.62), it follows from Eq. (6.59) that the probability density of ξ is

$$p_\xi(\xi) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\ \xi^{-\frac{1}{2}}\ e^{-\xi/2} & , \quad \xi > 0 \\ \\ 0 & , \quad \xi < 0 \ , \end{cases} \tag{6.64}$$

where we have used the fact that $(dx/d\xi) = \sigma^2$. Equation (6.64) is the usual normalized form of the chi-square density function with one degree-of-freedom. The moments of ξ are given on p. 405 of Ref. 20. See Eqs. (65) and (66) of Ref. 20, using $m = 1$. These moments are

$$M_\xi^{(n)} = \frac{\Gamma(\frac{1}{2}+n)}{\sqrt{\pi}}\ 2^n \quad , \tag{6.65}$$

where we have used the fact that $\Gamma(1/2) = \sqrt{\pi}$. Using the relationship $\Gamma(n+1) = n\ \Gamma(n)$, together with $\Gamma(1/2) = \sqrt{\pi}$, one may successively write out $\Gamma(\frac{1}{2}+n)$ for $n = 1,2,\ldots$ to discover that

$$\Gamma(\tfrac{1}{2}+n) = \frac{1\cdot3\cdot5\ \ldots\ (2n-1)}{2^n}\sqrt{\pi} \quad . \tag{6.66}$$

Combining Eqs. (6.63b), (6.65), and (6.66) yields the desired expression for $M_x^{(n)}$:

$$M_x^{(n)} = 1\cdot3\cdot5\ \ldots\ (2n-1)\sigma_{z_h}^{2n} \quad , \tag{6.67}$$

where we have inserted the subscript z_h on the σ to denote that it represents the standard deviation of the variable z_h. Combining Eqs. (6.56) and (6.67) and using the notation of Eq. (6.60) together with the definition

$$E[w_h^2] = \sigma_{z_h}^2 \quad , \tag{6.68}$$

we obtain for the nth moment of σ_f^2

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To put the above expression in a more useful form, we note that for n = 1, Eq. (6.69) becomes

$$E[\sigma_f^2] = \frac{E[w_h^2]}{(A'\sigma_{z_h})^2} \quad . \tag{6.70}$$

Solving Eq. (6.70) for $(A'\sigma_{z_h})^2$ and inserting that expression into Eq. (6.69) yields, finally,

$$E\{[\sigma_f^2]^n\} = M_{\sigma_f^2}^{(n)} = \left\{\frac{E[\sigma_f^2]}{E[w_h^2]}\right\}^n \frac{E\{[w_h^2]^n\}}{1\cdot 3\cdot 5 \ldots (2n-1)} \quad , \; n = 1,2,\ldots \; . \tag{6.71}$$

Equation (6.71) is the decired expression for the moments $M_{\sigma_f^2}^{(n)}$, *n = 1,2,3,... of* σ_f^2. *Equation (6.51) yields the central moments* $\mu_{\sigma_f^2}^{(n)}$ *from these moments. In evaluating the right-hand side of Eq. (6.71), the moments* $E\{[w_h^2]^n\}$, *n = 1,2,3,... are to be computed directly from the squared high-pass filtered turbulence sample* $w_h(t)$ *described by Eq. (6.13). The quantity* $E[\sigma_f^2]$ *can be determined from the method illustrated in Fig. 9 or the method described in Appendix G.*

Probability density function of σ_f^2. Let us now turn to estimation of the probability density of σ_f^2. Since we already have an expression, Eq. (6.71), for the moments of σ_f^2, it will be convenient to develop an approximation for the probability density function of σ_f^2 from these moments. To do so, we shall use an extension of the Gram-Charlier series.

The Gram-Charlier series is an expansion of a probability density function, the first term in the expansion being a normal probability density function with the correct mean and variance and the coefficients of the, say, N correction terms being chosen so that the *moments* through order N of the series approximation are set equal to the moments of the original distribution. The Gram-Charlier series is described in this manner on pp. 270-272 of Ref. 14. However, the Gram-Charlier series usually is

not derived using this moment equivalence criterion – e.g., Ref. 9, pp. 222-223 or Ref. 20, pp. 136-137. Using the moment criterion for choice of the correction term coefficients, this writer has extended the Gram-Charlier series to completely arbitrary "base density functions" (analogous to the normal probability density) on pp. 269-278 of Ref. 16 (where the application was to a different problem).

Since the random variable σ_f^2 under present discussion cannot be negative, the normal probability density function is not an appropriate "base density function" for expansion of σ_f^2. However, the gamma density function – e.g., pp. 220-221 of Ref. 10 – is appropriate since it has the flexibility required in the present application, yet does not permit negative values of the variate it describes. The gamma probability density has the form

$$p(V;\gamma,\lambda) = \begin{cases} \frac{\lambda}{\Gamma(\gamma)} (\lambda V)^{\gamma-1} e^{-\lambda V} & , \quad V > 0 \\ 0 & , \quad V < 0 \quad , \end{cases} \tag{6.72}$$

where $\Gamma(\gamma)$ is the gamma function and where $V = \sigma_f^2$ is the variate being described.

Notice that $p(V;\gamma,\lambda)$ contains two "free" parameters, γ and λ. The mean and variance of $p(V;\gamma,\lambda)$ are related to γ and λ by

$$E\{V\} = M_V^{(1)} = \frac{\gamma}{\lambda} \tag{6.73}$$

$$E[V^2] - \{E[V]\}^2 = M_V^{(2)} - (M_V^{(1)})^2 = \frac{\gamma}{\lambda^2} \quad , \tag{6.74}$$

from which we may solve for γ and λ in terms of the mean and variance to give

$$\gamma = \frac{(M_V^{(1)})^2}{M_V^{(2)} - (M_V^{(1)})^2} \tag{6.75}$$

$$\lambda = \frac{\gamma}{M_V^{(1)}} = \frac{M_V^{(1)}}{M_V^{(2)} - (M_V^{(1)})^2} \quad . \tag{6.76}$$

Of particular interest, in the present application, is the capability of $p(V;\gamma,\lambda)$ to approximate the density function of $V = \sigma_f^2$ when the relative standard deviation (or relative variance) of σ_f^2 is small – i.e., when the ratio of the standard deviation of σ_f^2 to the mean of σ_f^2 is small. According to Eq. (6.75), this ratio (for the variate V) is given by $\gamma^{-\frac{1}{2}}$. Thus, the asymptotic form of $p(V;\gamma,\lambda)$ for large γ is of interest. However, it is immediately evident from the form of the characteristic function of the gamma density $p(V;\gamma,\lambda)$ – e.g., p. 221 of Ref. 10 – that, according to the central limit theorem, the gamma density function approaches a normal or Gaussian density as $\gamma \to \infty$; furthermore, in the actual limit $\gamma = \infty$, $p(V;\gamma,\lambda)$ becomes a Dirac delta function located at $V = E\{V\} = M_V^{(1)}$, when γ and λ are chosen to satisfy Eqs. (6.75) to (6.76). Thus, for small values of relative standard deviation, $p(V;\gamma,\lambda)$ has the general appearance of a Gaussian density centered at $E\{V\}$, but with a truncated tail so that the density is zero for negative values of V. For large values of relative standard deviation, $p(V;\gamma,\lambda)$ can take on a considerable range of shapes, as an examination of Eq. (6.72) will indicate. Furthermore, it is of some interest to note that, if σ_f were to be exactly normally distributed with zero mean value (which we have ruled out by hypothesis), then the probability density of σ_f^2 would be exactly described by Eq. (6.72) with $\gamma = 1/2$.

From the above comments, we might expect Eq. (6.72) to provide a reasonable approximation to the probability density of $V = \sigma_f^2$ if the mean and variance of the density are chosen to take on the mean and variance of the variate $V = \sigma_f^2$, a decision which would require that γ and λ be chosen by Eqs. (6.75) and (6.76). Thus, computation of only two moments $M_{\sigma_f^2}^{(1)}$ and $M_{\sigma_f^2}^{(2)}$ by Eq. (6.71) should provide a reasonable first estimate of the probability density function of σ_f^2.

Let us now consider the extension of the Gram-Charlier series to the base density function given by Eq. (6.72). This extension is provided on pp. 272, 273, and 276 of Ref. 16, where in Eq. (118) of Ref. 16, we must set $A = \lambda/\Gamma(\gamma)$, since the probability density $p(V;\gamma,\lambda)$ must have unit area. It follows from Eqs. (97), (114), and (118) of Ref. 16 that our generalized form of the probability density of $V = \sigma_f^2$ can be written as

$$p_{\sigma_f^2}(V) = \begin{cases} \dfrac{\lambda}{\Gamma(\gamma)} (\lambda V)^{\gamma-1} e^{-\lambda V} \left[1 + \sum\limits_{n=3}^{N} b'_n \, p_n(V)\right], & V > 0 \\ \\ 0 & , \quad V < 0, \end{cases} \quad (6.77)$$

where

$$p_n(V) = {}_1F_1(-n;\gamma;\lambda V) \tag{6.78a}$$

$$= \sum_{k=0}^{n} \frac{(-n)_k \, \lambda^k}{k!(\gamma)_k} V^k \quad , \quad n = 1,2,3,\ldots \tag{6.78b}$$

are proportional to the generalized Laguerre polynomials, where ${}_1F_1(-n;\gamma;\lambda V)$ is the confluent hypergeometric function, and where, in Eq. (6.78b), we have used the notation

$$(u)_0 \triangleq 1$$

$$(u)_k \triangleq u(u+1) \ldots (u+k-1) \quad , \quad k \geq 1 \quad . \tag{6.79}$$

The expansion coefficients b_n' in Eq. (6.77) are related to those in Eq. (97) of Ref. 16 by

$$b_n' = \frac{b_n}{A} = \frac{\Gamma(\gamma)}{\lambda} b_n \quad , \tag{6.80}$$

as may be seen by examination of Eqs. (97), (114), (118), and (122) of Ref. 16, and by recognition of the fact that, in the present application, we must have $M^{(0)} = 1$. Consequently, by substitution of Eqs. (94b) and (117) of Ref. 16 into Eq. (6.80) above, we find for our expansion coefficients b_n',

$$b_n' = \frac{(\gamma)_n}{n!} \sum_{k=0}^{n} \alpha_{nk} \, M_{\sigma_f^2}^{(k)} \quad , \tag{6.81}$$

where, from Eq. (116) of Ref. 16, we have

$$\alpha_{nk} = \frac{(-n)_k \, \lambda^k}{k!(\gamma)_k} \quad , \tag{6.82}$$

which are the coefficients of the polynomials defined by Eq. (6.78b) above. In Eqs. (6.81) and (6.82), we have used again the notation of Eq. (6.79).

Equations (6.77), (6.78), (6.81), and (6.82) collectively describe the desired approximation to the probability density function of σ_f^2*, where the two parameters* γ *and* λ *are to be evaluated by Eqs. (6.75) and (6.76) from the first two moments of* $V = \sigma_f^2$*. The series in Eq. (6.77) can be truncated at any value of the index* N*. According to Eq. (6.81), for any integer value of* $N \geq 2$*, we can evaluate the expansion coefficients* b_n' *in Eq. (6.77) from the sequence of* N *moments* $M_{\sigma_f^2}^{(1)}$*,* $M_{\sigma_f^2}^{(2)}$*, ...,* $M_{\sigma_f^2}^{(N)}$ *that are to be determined by Eq. (6.71). Moreover, Eq. (6.77) has the property that we may add additional terms without changing the values of the coefficients of the terms previously determined. It has been shown in Ref. 16 that, for any value of* $N \geq 2$*, the moments* $M^{(1)}$*,* $M^{(2)}$*, ...,* $M^{(N)}$ *of the approximation* $f_{\sigma_f^2}(V)$ *given by the right-hand side of Eq. (6.77) are set equal to the moments* $M_{\sigma_f^2}^{(1)}$*,* $M_{\sigma_f^2}^{(2)}$*, ...,* $M_{\sigma_f^2}^{(N)}$*, when the expansion coefficients are determined by Eq. (6.81) and when* γ *and* λ *are determined by Eqs. (6.75) and (6.76).*

For values of V large relative to the mean $E[V] = \gamma/\lambda$, the tail in the approximation to $p_{\sigma_f^2}(V)$ given by Eq. (6.77) can go negative for $N > 2$. Some judgment will have to be used in choosing a value of N to prevent this occurrence and possibly other undesirable behavior of the series of Eq. (6.77). It is unlikely that much accuracy of any utility will be gained by using values of N larger than 4 (two correction terms in the right-hand side of Eq. 6.77), and, for practical purposes, the approximation given by Eq. (6.72) (no correction terms in the right-hand side of Eq. (6.77) will probably be adequate in most cases.

Estimation of Moments and Probability Density of $w_s(t)$

Moments of w_s. To form our estimate of the probability density $p_{w_s}(w_s)$ of $w_s(t)$, we shall use the moments of $w_s(t)$ to generate the coefficients of a Gram-Charlier expansion of $p_{w_s}(w_s)$. Moments will be used here because the moments of $w_s(t)$ are particularly easy to compute from the moments of $w_h(t)$ and $w(t)$. For the present application, we shall require the moments of $w_h(t)$, rather than the moments of its square $w_h^2(t)$.

According to the turbulence model of Eq. (2.3), the turbulence record $w(t)$ is the sum of the "slow" and "fast" processes $w_s(t)$ and $w_f(t)$; i.e.,

$$w(t) = w_s(t) + w_f(t) \quad . \tag{6.83}$$

The moments

$$M_w^{(n)} = E[w^n], \; n = 1,2,3,... \tag{6.84}$$

of the record $w(t)$ can be computed directly. Let us assume, for now that we have available the moments

$$M_{w_f}^{(n)} = E[w_f^n], \; n = 1,2,3,... \tag{6.85}$$

of the "fast" component $w_f(t)$. It is shown in Appendix H that the moments of the "slow" component

$$M_{w_s}^{(n)} = E[w_s^n], \; n = 1,2,3,... \tag{6.86}$$

can be computed sequentially from the sequences $M_w^{(n)}$, and $M_{w_f}^{(n)}$, $n = 1,2,3,...$ by the relationship

$$M_{w_s}^{(n)} = M_w^{(n)} - \sum_{k=0}^{n-1} \binom{n}{k} M_{w_f}^{(n-k)} \, M_{w_s}^{(k)}, \; n = 1,2,3,... \; , \tag{6.87}$$

where

$$M_{w_f}^{(0)} = M_{w_s}^{(0)} \equiv 1 \tag{6.88}$$

are the areas under the probability density functions of $w_f(t)$ and $w_s(t)$ and where $\binom{n}{k}$ are the binomial coefficients defined by Eq. (6.53). The first four of the sequence of the relationships of Eq. (6.87) are

$$M_{w_s}^{(1)} = M_w^{(1)} - M_{w_f}^{(1)}$$

$$M_{w_s}^{(2)} = M_w^{(2)} - [M_{w_f}^{(2)} + 2M_{w_f}^{(1)} M_{w_s}^{(1)}]$$

$$M_{w_s}^{(3)} = M_w^{(3)} - [M_{w_f}^{(3)} + 3M_{w_f}^{(2)} M_{w_s}^{(1)} + 3M_{w_f}^{(1)} M_{w_s}^{(2)}]$$

$$M_{w_s}^{(4)} = M_w^{(4)} - [M_{w_f}^{(4)} + 4M_{w_f}^{(3)} M_{w_s}^{(1)} + 6M_{w_f}^{(2)} M_{w_s}^{(2)} + 4M_{w_f}^{(1)} M_{w_s}^{(3)}].$$

(6.89)

Notice from Eqs. (6.87) and (6.89) that to compute the moments through order N of $w_s(t)$, only the moments through order N of $w(t)$ and $w_f(t)$ are required.

The moments $M_{w_f}^{(n)}$, n = 1,2,3,... of the component $w_f(t)$ are to be computed from the high-pass filtered version $w_h(t)$ of the turbulence record $w(t)$, where this high-pass filtered waveform $w_h(t)$ was discussed earlier in connection with Eq. (6.13). It is shown in Appendix G that the moments $M_{w_f}^{(n)}$, n = 1,2,3,... can be computed from the moments $M_{w_h}^{(n)}$, n = 1,2,3,... of the record $w_h(t)$ by

$$M_{w_f}^{(n)} = K^n M_{w_h}^{(n)}, \quad n = 1,2,3,\ldots \quad , \tag{6.90}$$

where the moments $M_{w_h}^{(n)}$ are, of course, defined by the relationship

$$M_{w_h}^{(n)} \triangleq E[w_h^n] \;, \quad n = 1,2,3,\ldots \quad . \tag{6.91}$$

The positive constant K in Eq. (6.90) is shown in Appendix G to be given by

$$K = \left[\frac{\int_{-\infty}^{\infty} S_z(f)df}{\int_{-\infty}^{\infty} |H_h(f)|^2 \, S_z(f)df} \right]^{\frac{1}{2}} \quad , \tag{6.92}$$

where $S_z(f)$ is the power spectral density of the component $z(t)$ in the model of Eq. (2.3) and where $H_h(f)$ is the high-pass filter complex frequency-response function. As we described in Sec. 6.1, we shall generally want to assume that $S_z(f)$ is the appropriate von Karman spectral form. Moreover, since $E[z^2] = 1$ [according to Eq. (2.3)], the numerator in the right-hand side of Eq. (6.92) will be unity.

Once the integral scale of $z(t)$ has been determined and the high-pass filter has been chosen, the constant K can be computed. Furthermore, from a turbulence record $w(t)$, we may generate the moments $M_w^{(n)}$, and the high-pass filtered record $w_h(t)$, from which we may compute its moments $M_{w_h}^{(n)}$. From these moments and the constant K, we can use Eq. (6.90) to compute the moments $M_{w_f}^{(n)}$, which we may then combine with the moments $M_w^{(n)}$, using Eqs. (6.87) or (6.89), to compute the moments $M_s^{(n)}$.

Probability density function of w_s. To generate an estimate of the probability density of $w_s(t)$ using the Gram-Charlier expansion, we must use the *central moments* of $w_s(t)$. If $w(t)$ has zero mean value, then it is evident from the first of the four equations in (6.89) that we should have $M_{w_s}^{(1)} = 0$; i.e., when $M_{w_s}^{(1)} = 0$, the moments $M_{w_s}^{(n)}$, $n = 1,2,3,\ldots$, are the *central* moments. We shall assume that $M_{w_s}^{(1)} \equiv 0$ in the following discussion.

It is shown on pp. 270-272 of Ref. 14 that the Gram-Charlier expansion of a probability density function, say, $p_{w_s}(w_s)$, may be expressed as

$$p_{w_s}(w_s) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{w_s^2}{2\sigma^2}} \left[1 + \frac{M_{w_s}^{(3)}}{3!\sigma^3}\left(\frac{w_s^3}{\sigma^3} - \frac{3w_s}{\sigma}\right) + \frac{1}{4!}\left(\frac{M_{w_s}^{(4)}}{\sigma^4} - 3\right)\left(\frac{w_s^4}{\sigma^4} - \frac{6w_s^2}{\sigma^2} + 3\right) + \ldots \right], \tag{6.93}$$

where, here,

$$\sigma = \{E[w_s^2]\}^{1/2} = [M_{w_s}^{(2)}]^{1/2} \tag{6.94}$$

and where only two correction terms have been retained, the first being an odd function of w_s and the second being an even function

of w_s. These two correction terms are adequate to determine how closely $p_{w_s}(w_s)$ conforms to a Gaussian probability density function. It may be shown that the results provided on pp. 277-278 of Ref. 16 also lead to the result given by Eq. (6.93).

To evaluate the parameters in Eq. (6.93), one requires $\sigma = [M_{w_s}^{(2)}]^{\frac{1}{2}}$, $M_{w_s}^{(3)}$, and $M_{w_s}^{(4)}$. These quantities are to be evaluated using Eq. (6.89).

APPENDIX A

DERIVATION OF REQUIREMENT FOR NEGLIGIBLY SMALL CORRELATION COEFFICIENT BETWEEN A NONSTATIONARY PROCESS AND ITS DERIVATIVE

Consider an arbitrary function $y(t)$ and its derivative $y'(t)$. Integrating the product of y and y' by parts, we find

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} y(\xi)y'(\xi)d\xi = \frac{1}{\Delta t}\left[y^2(\xi)\Big|_t^{t+\Delta t} - \int_t^{t+\Delta t} y(\xi)\ y'(\xi)d\xi\right]$$

$$= \frac{1}{\Delta t}\left[y^2(t+\Delta t) - y^2(t) - \int_t^{t+\Delta t} y(\xi)\ y'(\xi)d\xi\right]; \tag{A.1}$$

hence, we have, exactly,

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} y(\xi)\ y'(\xi)d\xi = \frac{1}{2}\,\frac{y^2(t+\Delta t) - y^2(t)}{\Delta t}\ . \tag{A.2}$$

Let us now assume that $\{y(t)\}$ is a generally nonstationary stochastic process. Taking the expected value of Eq. (A.2) and then interchanging the order of expectation and integration operations on the left-hand side gives

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} E\{y(\xi)\ y'(\xi)\}d\xi = \frac{1}{2}\,\frac{E\{y^2(t+\Delta t)\} - E\{y^2(t)\}}{\Delta t}\ . \tag{A.3}$$

If we now take the limit $\Delta t \to 0$ in Eq. (A.3), we have, assuming that $E\{y(\xi)\ y'(\xi)\}$ is continuous,

$$E\{y(t)\ y'(t)\} = \frac{1}{2}\,\frac{d}{dt}\,E\{y^2(t)\}\ , \tag{A.4}$$

which holds for nonstationary and stationary processes. For wide-sense stationary processes, the right-hand side of Eq. (A.4) is zero; hence, for wide-sense stationary processes, Eq. (A.4) reduces to the usual result $E\{y(t)\ y'(t)\} = 0$.

Denoting by $\sigma_y^2(t)$ and $\sigma_{y'}^2(t)$ the mean-square values of $y(t)$ and $y'(t)$ and assuming that

$$E\{y(t)\} = 0 \quad , \quad \text{for all } t \quad , \tag{A.5}$$

we may express the correlation coefficient of $y(t)$ and $y'(t)$ as

$$\rho_{y,y'}(t) \triangleq \frac{E\{y(t)\ y'(t)\}}{\sigma_y(t)\ \sigma_{y'}(t)}$$

$$= \frac{1}{2}\ \frac{\sigma_y(t)}{\sigma_{y'}(t)}\ \frac{\frac{d}{dt}\ \sigma_y^2(t)}{\sigma_y^2(t)}$$

$$= \frac{1}{2}\ \frac{\sigma_y(t)}{\sigma_{y'}(t)}\ \frac{d}{dt}\ \ell n\ \sigma_y^2(t) \quad , \tag{A.6}$$

which is a completely general result for nonstationary processes.

At this juncture, we shall assume that the locally stationary conditions of Eqs. (3.41), (3.43), and (3.46) are satisfied; hence, the response-process conditional instantaneous spectrum is well approximated by Eq. (3.40). When these conditions are met, we shall show that the right-hand side of Eq. (A.6) is negligibly small in comparison with unity. In carrying out the heuristic proof to follow, we shall further assume that the "slow" component of excitation $w_s(t)$ in Eq. (2.3) is zero, which implies that $\Phi_{w_s}(f)$ in Eq. (3.40) is zero. Removal of this stationary component of the response process has the effect of increasing the magnitude of the right-hand side of Eq. (A.6), which is zero for stationary processes; hence, the assumption that $\Phi_{w_s}(f)$ is zero is conservative.

Integrating Eq. (3.40) over $-\infty < f < \infty$ and using a fundamental property of the instantaneous spectrum – e.g., Eq. (12a) of Ref. 7 or Eq. (2.4a) of Ref. 6 – we have

$$\sigma_y^2(t) = \int_{-\infty}^{\infty} \Phi_y(f,t\,|\,\sigma_f)df$$

$$\approx \sigma_f^2(t)\sigma_{y_z}^2 \quad , \tag{A.7}$$

where we have set $\Phi_{w_s}(f)$ equal to zero in Eq. (3.40) before carrying out the above integration, and where $\Phi^2_{y_z}$ is the mean-square aircraft response to the stationary component $z(t)$ of the "fast" turbulence component $\sigma_f(t)\ z(t)$ in Eq. (2.3); i.e.,

$$\sigma^2_{y_z} \triangleq \int_{-\infty}^{\infty} \Phi_z(f)\ |H(f)|^2 df \quad . \tag{A.8}$$

Equation (A.7) is the desired expression for $\sigma^2_y(t)$ for use in Eq. (A.6). Let us now obtain an expression for $\sigma^2_{y'}(t)$. The locally stationary response approximation of Eq. (3.40) implies that we may write the response process as

$$y(t) \approx \sigma_f(t)\ y_z(t) \quad , \tag{A.9}$$

where $y_z(t)$ is the aircraft response to the component $z(t)$ of Eq. (2.3). Differentiating Eq. (A.9) gives

$$y'(t) = \sigma_f(t)\ y'_z(t) + \sigma'_f(t)\ y_z(t) \quad ; \tag{A.10}$$

hence,

$$\begin{aligned}\sigma^2_{y'}(t) &\triangleq E\{[y'(t)]^2\} \\ &= \sigma^2_f(t)\ E\{[y'_z]^2\} + 2\sigma_f(t)\sigma'_f(t)\ E\{y'_z y_z\} + [\sigma'_f(t)]^2 E\{y^2_z\} \\ &= \sigma^2_f(t)\ \sigma^2_{y'_z} + [\sigma'_f(t)]^2\ \sigma^2_{y_z} \quad ,\end{aligned} \tag{A.11}$$

where the middle term in the second line of the above equation is zero because $E\{y'_z y_z\} = 0$, which follows from the fact that the process $\{y_z(t)\}$ is stationary. Now, for locally stationary processes that satisfy Eq. (3.40), we would expect the second term in Eq. (A.11) to be small in comparison with the first term. Neglecting the second term yields the locally stationary approximation to $\sigma^2_{y'}(t)$ — i.e.,

$$\sigma^2_{y'}(t) \approx \sigma^2_f(t)\ \sigma^2_{y'_z} \quad ; \tag{A.12}$$

hence, from Eqs. (A.7) and (A.12), we have

$$\frac{\sigma_y(t)}{\sigma_{y'}(t)} \approx \frac{\sigma_{y_z}}{\sigma_{y'_z}} \quad , \tag{A.13}$$

which is independent of time. Moreover, we note that the approximation given by the right-hand side of Eq. (A.12) is always less than the right-hand side of Eq. (A.11); hence, the approximation of Eq. (A.12) has the effect of increasing the size of the right-hand side of Eq. (A.6); i.e., insofar as the present discussion is concerned, the approximation of Eq. (A.12) is conservative. We may summarize these results as

$$\rho_{y,y'}(t) \approx \frac{1}{2} \frac{\sigma_{y_z}}{\sigma_{y'_z}} \frac{d}{dt} \ln \sigma_y^2(t) \quad , \tag{A.14}$$

where the approximation of Eq. (A.14) is valid whenever Eq. (A.13) is valid, which requires that the response to the "slow" component of turbulence $w_s(t)$ be negligible in comparison with the response to the "fast" component $\sigma_f(t)\, z(t)$, and furthermore that Eq. (3.40) be satisfied.

Let us now consider the ratio $\sigma_{y_z}/\sigma_{y'_z}$. It is well known – e.g., Ref. 11, pp. 190-192 of the Wax edition or Ref. 12, p. 48 – that $\sigma_{y_z}/\sigma_{y'_z}$ can be expressed in terms of the autocorrelation function $\phi_{y_z}(\tau)$ of $y_z(t)$ by

$$\frac{\sigma_{y_z}}{\sigma_{y'_z}} = \left[- \frac{\phi_{y_z}(0)}{\phi''_{y_z}(0)} \right]^{1/2} \quad . \tag{A.15}$$

The right-hand side of Eq. (A.15) has a simple interpretation that is analogous to the definition of the Taylor microscale; e.g., Ref. 19, p. 42. Let us approximate $\phi_{y_z}(\tau)$ by the parabola $\phi_{y_z}(\tau)_p$:

$$\phi_{y_z}(\tau)_p \triangleq \phi_{y_z}(0) \left[1 - \frac{\tau^2}{\tau_0^2} \right] \quad . \tag{A.16}$$

It follows directly that the parabolic approximation to $\phi_{y_z}(\tau)$ given by Eq. (A.16) becomes zero at $\tau = \pm\tau_0$. However, differentiating Eq. (A.16) twice, we find

$$\phi''_{y_z}(\tau)_p = -\frac{2\phi_{y_z}(0)}{\tau_0^2} \quad . \tag{A.17}$$

Hence, we have

$$\left[-\frac{\phi_{y_z}(0)}{\phi''_{y_z}(0)_p}\right]^{\frac{1}{2}} = \frac{\tau_0}{\sqrt{2}} = 0.71\tau_0 \quad ; \tag{A.18}$$

that is, when a parabolic approximation to $\phi_{y_z}(\tau)$ is used, the right-hand side of Eq. (A.15) is slightly less than the time delay associated with the zero crossing of $\phi_{y_z}(\tau)_p$. In the turbulence application, τ_0 is the analog of the Taylor microscale.

Equation (A.15) has another interpretation. It follows directly from pp. 192-193 of the Wax edition of Ref. 11 or from Eq. (1.65) of Ref. 12 that $(\sigma_{y_z}/\sigma_{y'_z})$ is equal to $(1/\pi)$ times the expected time between zero crossings of the Gaussian process $\{y_z(t)\}$. Now, we generally would expect τ_0 to be about one-half of the nominal "correlation time" of the process. *Consequently, we may conclude from both of the above interpretations that* $(\sigma_{y_z}/\sigma_{y'_z})$ *typically is of the order of one-third of the nominal "correlation interval" of* τ_{cor} *of the process* $y_z(t)$.

Using the above interpretation of $\sigma_{y_z}/\sigma_{y'_z}$, we may now interpret the right-hand side of Eq. (A.14). Letting the symbol ~ denote "is of the order of," we have

$$\rho_{\dot{y},y'}(t) \sim \frac{1}{6}\,\tau_{cor}\,\frac{\frac{d}{dt}\,\sigma_y^2(t)}{\sigma_y^2(t)} \quad , \tag{A.19}$$

i.e., $\rho_{y,y'}(t)$ *is of the order of one-sixth of the fractional change of* $\sigma_y^2(t)$ *that occurs in one nominal correlation interval of the process* $y_z(t)$. In the present application, the fractional change in $\sigma_y^2(t)$ is approximately equal to the fractional change

in $\sigma_f^2(t)$ - see Eq. (A.7) and recall that $\sigma_{y_z}^2$ is a constant. Hence, we conclude that when the fractional change in $\sigma_f^2(t)$ is negligible over the nominal correlation interval of the response process, we have $\rho_{y,y'}(t) \approx 0$. Finally, we note that the requirement that the fractional change in $\sigma_f^2(t)$ over the correlation interval of the response process be negligible is essentially equivalent to the requirement for the validity of the quasi-stationary approximation of Eq. (3.40), as is evident from the discussion in Sec. 5 of this report. *Thus, whenever the approximation of Eq. (3.40) is valid, it is permissible to assume that the correlation coefficient between $y(t)|\sigma_f$ and its first derivative is negligibly small.*

APPENDIX B

DERIVATION OF CORRECTION TERM TO GAUSSIAN APPROXIMATION OF EXCEEDANCE PLOTS

To develop an expression for the correction term $Q^{(2)}(y|\sigma_f^2)$ in Eq. (4.36), we require, according to Eqs. (4.33) and (4.34), an expression for the second derivative with respect to σ_f^2 of $N_+(y|\sigma_f^2)$. See Eq. (4.26). According to Eqs. (4.12), (4.14), and (4.15), $N_+(y|\sigma_f^2)$ can be expressed as

$$N_+(y|\sigma_f^2) = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{y}}^2}{\sigma_y^2}\right)^{\frac{1}{2}} e^{-\frac{y^2}{2\sigma_y^2}} , \tag{B.1}$$

where, according to Eqs. (4.16) and (4.19), we have

$$\sigma_y^2 = \sigma_{y_s}^2 + \sigma_f^2\,\sigma_{y_z}^2 \tag{B.2}$$

and

$$\sigma_{\dot{y}}^2 = \sigma_{\dot{y}_s}^2 + \sigma_f^2\,\sigma_{\dot{y}_z}^2 , \tag{B.3}$$

and where the explicit dependence of σ_y^2 and $\sigma_{\dot{y}}^2$ on σ_f^2 has been deleted in the notation of Eq. (B.1) and in the left-hand sides of Eqs. (B.2) and (B.3).

Forming the first derivative of $N_+(y|\sigma_f^2)$ with respect to σ_f^2, we have, using Eq. (B.1) and the notation of Eq. (4.26),

$$N_+^{(1)}(y|\sigma_f^2) = \frac{d}{d\sigma_f^2} N_+(y|\sigma_f^2) \quad .$$

Carrying out the differentiation yields

$$N_+^{(1)}(y|\sigma_f^2) = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{y}}^2}{\sigma_y^2}\right)^{\frac{1}{2}} e^{-\frac{y^2}{2\sigma_y^2}}\left[\frac{y^2}{2(\sigma_y^2)^2}\,\frac{d\sigma_y^2}{d\sigma_f^2}\right]$$

$$+ \frac{1}{2\pi}\, e^{-\frac{y^2}{2\sigma_y^2}}\,\frac{1}{2}\left(\frac{\sigma_{\dot{y}}^2}{\sigma_y^2}\right)^{-\frac{1}{2}}\left[\frac{\sigma_y^2\,\frac{d\sigma_{\dot{y}}^2}{d\sigma_f^2} - \sigma_{\dot{y}}^2\,\frac{d\sigma_y^2}{d\sigma_f^2}}{(\sigma_y^2)^2}\right]$$

$$= \frac{1}{2}\, N_+(y|\sigma_f^2)\left[\frac{y^2}{\sigma_y^2}\,\frac{\frac{d\sigma_y^2}{d\sigma_f^2}}{\sigma_y^2} + \frac{\frac{d\sigma_{\dot{y}}^2}{d\sigma_f^2}}{\sigma_{\dot{y}}^2} - \frac{\frac{d\sigma_y^2}{d\sigma_f^2}}{\sigma_y^2}\right]$$

$$= \frac{1}{2}\, N_+(y|\sigma_f^2)\left[\frac{\frac{d\sigma_y^2}{d\sigma_f^2}}{\sigma_y^2}\left(\frac{y^2}{\sigma_y^2} - 1\right) + \frac{\frac{d\sigma_{\dot{y}}^2}{d\sigma_f^2}}{\sigma_{\dot{y}}^2}\right], \tag{B.4}$$

where we have used the definition of Eq. (B.1). But from Eqs. (B.2) and (B.3), we have

$$\frac{d\sigma_y^2}{d\sigma_f^2} = \sigma_{y_z}^2 \qquad \frac{d\sigma_{\dot{y}}^2}{d\sigma_f^2} = \sigma_{\dot{y}_z}^2 \quad ; \tag{B.5a,b}$$

hence, Eq. (B.4) can be expressed as

$$N_+^{(1)}(y|\sigma_f^2) = \frac{1}{2}\, N_+(y|\sigma_f^2)\left[\frac{\sigma_{y_z}^2}{\sigma_y^2}\left(\frac{y^2}{\sigma_y^2} - 1\right) + \frac{\sigma_{\dot{y}_z}^2}{\sigma_{\dot{y}}^2}\right]. \tag{B.6}$$

Differentiating Eq. (B.6) with respect to σ_f^2 yields

$$N_+^{(2)}(y|\sigma_f^2) = \frac{1}{2} N_+(y|\sigma_f^2) \left[\frac{\sigma^2_{y_z}}{\sigma_y^2}\left(-\frac{y^2}{(\sigma_y^2)^2}\frac{d\sigma_y^2}{d\sigma_f^2}\right)\right.$$

$$\left. - \left(\frac{y^2}{\sigma_y^2} - 1\right)\frac{\sigma^2_{y_z}}{(\sigma_y^2)^2}\frac{d\sigma_y^2}{d\sigma_f^2} - \frac{\sigma^2_{\dot{y}_z}}{(\sigma_{\dot{y}}^2)^2}\frac{d\sigma_{\dot{y}}^2}{d\sigma_f^2}\right]$$

$$+ \frac{1}{4}\left[\frac{\sigma^2_{y_z}}{\sigma_y^2}\left(\frac{y^2}{\sigma_y^2} - 1\right) + \frac{\sigma^2_{\dot{y}_z}}{\sigma_{\dot{y}}^2}\right]^2 N_+(y|\sigma_f^2). \tag{B.7}$$

Substituting Eqs. (B.5a,b) into Eq. (B.7), and simplifying the resulting expression gives, finally,

$$N_+^{(2)}(y|\sigma_f^2) = \frac{1}{2} N_+(y|\sigma_f^2) \left\{\frac{1}{2}\left[\frac{\sigma^2_{y_z}}{\sigma_y^2}\left(\frac{y^2}{\sigma_y^2} - 1\right) + \frac{\sigma^2_{\dot{y}_z}}{\sigma_{\dot{y}}^2}\right]^2\right.$$

$$\left. - \left[\frac{(\sigma^2_{y_z})^2}{(\sigma_y^2)^2}\left(2\frac{y^2}{\sigma_y^2} - 1\right) + \frac{(\sigma^2_{\dot{y}_z})^2}{(\sigma_{\dot{y}}^2)^2}\right]\right\} \tag{B.8}$$

When $\sigma_y^2|\sigma_f^2$ and $\sigma_{\dot{y}}^2|\sigma_f^2$ are written for σ_y^2 and σ_f^2 in Eq. (B.8), and the result is combined with Eq. (4.33), we obtain the definition of $Q^{(2)}(y|\sigma_f^2)$ given by Eq. (4.34).

APPENDIX C

DERIVATION OF CORRECTION TERM TO GAUSSIAN APPROXIMATION OF PROBABILITY DENSITY FUNCTION

According to Eqs. (4.49) through (4.51), the second derivative of $p(y|\sigma_f^2)$ with respect to σ_f^2 is required to derive an expression for the correction term $U^{(2)}(y|\sigma_f^2)$ to the Gaussian probability density $p(y|\overline{\sigma_f^2})$. Suppressing the dependence of σ_y^2 on σ_f^2, as given by Eq. (B.1), we have from Eq. (4.4) for $p(y|\sigma_f^2)$,

$$p(y|\sigma_f^2) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{y^2}{2\sigma_y^2}} \quad . \tag{C.1}$$

From the fact that $(d\sigma_y^2/d\sigma_f^2) = \sigma_{y_z}^2$ [see Eq. (B.5a)], it follows from Eq. (C.1) that the first derivative of $p(y|\sigma_f^2)$ with respect to σ_f^2 is

$$p^{(1)}(y|\sigma_f^2) = \frac{1}{\sqrt{2\pi}} \left[(\sigma_y^2)^{-\frac{1}{2}} e^{-\frac{y^2}{2\sigma_y^2}} \frac{y^2}{2(\sigma_y^2)^2} \sigma_{y_z}^2 - \frac{1}{2} e^{-\frac{y^2}{2\sigma_y^2}} (\sigma_y^2)^{-3/2} \sigma_{y_z}^2 \right]$$

$$= \frac{\sigma_{y_z}^2}{2} p(y|\sigma_f^2) \left[\frac{y^2}{(\sigma_y^2)^2} - \frac{1}{\sigma_y^2} \right] \quad . \tag{C.2}$$

Differentiating Eq. (C.2) with respect to σ_f^2, and again using Eq. (B.5a), we have

$$p^{(2)}(y|\sigma_f^2) = \frac{\sigma_{y_z}^2}{2} \left\{ p(y|\sigma_f^2) \left[-2y^2(\sigma_y^2)^{-3}\sigma_{y_z}^2 + (\sigma_y^2)^{-2}\sigma_{y_z}^2 \right] + \frac{\sigma_{y_z}^2}{2} \left[y^2(\sigma_y^2)^{-2} - (\sigma_y^2)^{-1} \right]^2 p(y|\sigma_f^2) \right\}$$

$$= \frac{1}{2} p(y|\sigma_f^2) \frac{(\sigma_{y_z}^2)^2}{(\sigma_y^2)^2} \left[1 - \frac{2y^2}{\sigma_y^2} + \frac{1}{2} \left(\frac{y^2}{\sigma_y^2} - 1 \right)^2 \right] . \tag{C.3}$$

(We remind the reader that we have used the notation σ_y^2 for $\sigma_y^2|\sigma_f^2$.) By combining Eqs. (4.49) and (C.3), we obtain the definition of $U^{(2)}(y|\sigma_f^2)$ given by Eq. (4.50).

APPENDIX D

DERIVATION OF EXPRESSION FOR COEFFICIENT OF EXCESS OF RESPONSE PROCESS

To show that the coefficient of excess $\gamma_y^{(2)}$ of the response process y(t) is given by Eq. (4.54), we first note that the mean-square response $E\{y^2\}$ is obtained from Eq. (4.35a) by

$$
\begin{aligned}
E\{y^2\} = \sigma_y^2 &= \int_0^\infty (\sigma_y^2|\sigma_f^2)\; p(\sigma_f^2)d\sigma_f^2 \\
&= \sigma_{y_s}^2 + \left[\int_0^\infty \sigma_f^2\; p(\sigma_f^2)d\sigma_f^2\right]\sigma_{y_z}^2 \\
&= \sigma_{y_s}^2 + \overline{\sigma_f^2}\,\sigma_{y_z}^2 \quad . \qquad \text{(D.1)}
\end{aligned}
$$

The fourth moment of the response y(t) may be expressed as

$$E\{y^4\} = \int_0^\infty E\{y^4|\sigma_f^2\}\; p(\sigma_f^2)d\sigma_f^2 \quad . \qquad \text{(D.2)}$$

Substituting Eq. (4.6) into Eq. (D.2), then using the expression for $\sigma_y^2|\sigma_f^2$ given by Eq. (4.35a), and then simplifying, we have

$$
\begin{aligned}
E\{y^4\} &= 3\int_0^\infty (\sigma_y^2|\sigma_f^2)^2\; p(\sigma_f^2)d\sigma_f^2 \\
&= 3\int_0^\infty (\sigma_{y_s}^2+\sigma_f^2\,\sigma_{y_z}^2)^2\; p(\sigma_f^2)d\sigma_f^2 \\
&= 3\int_0^\infty (\sigma_{y_s}^4+2\sigma_{y_s}^2\sigma_{y_z}^2\sigma_f^2 + \sigma_{y_z}^4\sigma_f^4)\; p(\sigma_f^2)d\sigma_f^2 \\
&= 3(\sigma_{y_s}^4 + 2\sigma_{y_s}^2\sigma_{y_z}^2\overline{\sigma_f^2} + \sigma_{y_z}^4\overline{\sigma_f^4}) \quad . \qquad \text{(D.3)}
\end{aligned}
$$

Noting that $\mu_y^{(4)}$ is the fourth central moment of y and that the mean value of y is zero, we can see from Eqs. (4.52), (D.3), and (D.1) that the coefficient of excess of y may be expressed by

$$\gamma_y^{(2)} =$$

$$= \frac{3[(\sigma^4_{y_s} + 2\sigma^2_{y_s}\sigma^2_{y_z}\overline{\sigma^2_f} + \sigma^4_{y_z}\overline{\sigma^4_f}) - (\sigma^4_{y_s} + 2\sigma^2_{y_s}\overline{\sigma^2_f}\sigma^2_{y_z} + \overline{\sigma^2_f}^2\sigma^4_{y_z})]}{\sigma^4_y}$$

$$= \frac{3\,\sigma^4_{y_z}(\overline{\sigma^4_f} - \overline{\sigma^2_f}^2)}{\sigma^4_y}$$

$$= \frac{3(\sigma^2_{y_z})^2\, E\{(\sigma^2_f)^2 - [E(\sigma^2_f)]^2\}}{(\sigma^2_y)^2}$$

$$= 3\,\frac{(\sigma^2_{y_z})^2}{(\sigma^2_y)^2}\,\mu^{(2)}_{\sigma^2_f} \quad , \qquad \text{(D.4)}$$

which is the expression given by Eq. (4.54), when it is recognized that $(\sigma^2_y|\overline{\sigma^2_f}) = \sigma^2_y$.

APPENDIX E

DERIVATION OF EQUATION (5.19)

To establish the validity of Eq. (5.19), we shall first show that for ergodic processes $v'(t)$, we have

$$\mathrm{Var}\{v''(t)+[v'(t)]^2\} = E\{[v''(t)]^2\} + \mathrm{Var}\{[v'(t)]^2\}. \tag{E.1}$$

Let us define

$$x(t) \triangleq v''(t) + [v'(t)]^2 . \tag{E.2}$$

Using a well known result, we have

$$\begin{aligned}
\mathrm{Var}\{v''(t)+[v'(t)]^2\} &= \mathrm{Var}\{x\} = E\{x^2\} - E^2\{x\} \\
&= E\{[v''+(v')^2]^2\} - (E\{v''\}+E\{(v')^2\})^2 \\
&= E\{(v'')^2\} + 2E\{v''(v')^2\} + E\{(v')^4\} \\
&\quad - E^2\{v''\} - 2E\{v''\}E\{(v')^2\} - E^2\{(v')^2\} \\
&= \mathrm{Var}\{v''\} + \mathrm{Var}\{(v')^2\} \\
&\quad + 2(E\{v''(v')^2\} - E\{v''\}E\{(v')^2\})
\end{aligned} \tag{E.3}$$

However, $v(t)$ is a stationary process; hence, $E\{v''\} = 0$. Using this result, we may write Eq. (E.3) as

$$\mathrm{Var}\{v''(t)+[v'(t)]^2\} = E\{(v'')^2\} + \mathrm{Var}\{(v')^2\} + 2(E\{v''(v')^2\}). \tag{E.4}$$

To evaluate the last term in the right-hand side of Eq. (E.4), we shall assume that $\{v'(t)\}$ is an ergodic process; hence, we have

$$E\{v''(v')^2\} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v''(t)[v'(t)]^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T} [v'(t)]^3 \Big|_{-T/2}^{T/2} - 2 \int_{-T/2}^{T/2} v''(t)[v'(t)]^2 dt \quad ,(E.5)$$

where the second line was obtained by integration by parts. It follows from Eq. (E.5) that

$$\frac{1}{T} \int_{-T/2}^{T/2} v''(t)[v'(t)]^2 dt = \frac{1}{3T} \{[v'(T/2)]^3 - [v'(-T/2)]^3\} . \quad (E.6)$$

However, for a stationary process $v'(t)$, the quantities $v'(T/2)$ and $v'(-T/2)$ are bounded; hence, the right-hand side of Eq. (E.6) vanishes in the limit $T\to\infty$. Consequently, we have

$$E\{v''(v')^2\} = 0. \quad (E.7)$$

Therefore, from Eq. (E.4), we have

$$\mathrm{Var}\{v''(t)+[v'(t)]^2\} = E\{[v''(t)]^2\} + \mathrm{Var}\{[v'(t)]^2\}, \quad (E.8)$$

which is the result given by Eq. (E.1).

Unfortunately, the quantity $\mathrm{Var}\{[v'(t)]^2\}$ depends on fourth order statistics of the process $v'(t)$. These statistics are impossible to evaluate from the autocorrelation function of $v(t)$, unless it is assumed that $v'(t)$ is a (stationary) Gaussian process, which necessarily has zero mean value because the process $\{v(t)\}$ is, by definition, stationary. If this Gaussian assumption is made, it is known (e.g., Ref. 21, p. 92) that

$$\mathrm{Var}\{[v'(t)]^2\} = 2E^2\{[v'(t)]^2\} \quad (E.9a)$$

$$= 2[R_v''(0)]^2 , \quad (E.9b)$$

where the second line is a consequence of Eq. (5.15). Furthermore, from Eq. (5.16), we have

$$E\{[v''(t)]^2\} = R_v^{(4)}(0). \quad (E.10)$$

By combining Eqs. (E.8), (E.9), and (E.10), we have, finally,

$$\mathrm{Var}\{v''(t)+[v'(t)]^2\} = R_V^{(4)}(0) + 2[R_V''(0)]^2, \qquad \text{(E.11)}$$

which is the result given by Eq. (5.19), and is strictly valid only for cases where $\{v'(t)\}$ is an ergodic Gaussian process.

APPENDIX F

DERIVATION OF RELATIONSHIP BETWEEN AUTOCORRELATION FUNCTION OF A STATIONARY GAUSSIAN PROCESS AND AUTOCORRELATION FUNCTION OF THE SQUARE OF ITS LOGARITHM

Here, we shall derive an expression for the autocorrelation function of $\ell n\ z_h^2(t)$; i.e.,

$$R(\tau) = E\{\ell n\ z_h^2(t)\ \ell n\ z_h^2(t+\tau)\} \tag{F.1}$$

in terms of the autocorrelation coefficient of the stationary Gaussian process $z_h(t)$, where

$$E\{z_h\} = 0 \quad , \quad E\{z_h^2\} = 1 \quad ; \tag{F.2a,b}$$

i.e.,

$$\rho(\tau) = E\{z_h(t)\ z_h(t+\tau)\} \quad . \tag{F.3}$$

We shall carry out the derivation using the form of Price's Theorem given by Eqs. (20) and (21) of Ref. 22.

Let $z_h(t)$ be a stationary Gaussian process satisfying Eqs. (F.2a,b), and let $f[z_h]$ be an arbitrary zero-memory (generally nonlinear) transformation of z_h. The form of the theorem that we shall use states that

$$\frac{\partial R}{\partial \rho} = E\{f'[z_h(t)]\ f'[z_h(t+\tau)]\} \quad , \tag{F.4}$$

where the primes denote the derivative of $f[z_h]$ with respect to z_h, and where these derivatives are evaluated at times t and $t + \tau$, as indicated. In the left-hand side, ρ denotes the correlation coefficient defined by Eq. (F.3) and R denotes

$$R(\tau) = E\{f[z_h(t)]\ f[z_h(t+\tau)]\} \quad . \tag{F.5}$$

Thus, comparing Eqs. (F.1) and (F.5), it is evident that we are interested in the case

$$f[z_h] = \ell n\ z_h^2 \quad ; \tag{F.6}$$

hence, we have

$$f'[z_h] = \frac{2}{z_h} \quad . \tag{F.7}$$

Substitution of Eq. (F.7) into Eq. (F.4) yields, when the expression for the joint Gaussian density is written out and where we substitute $z_1 = z_h(t)$ and $z_2 = z_h(t+\tau)$,

$$\frac{\partial R}{\partial \rho} = \frac{4}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{z_1 z_2}\, e^{-\frac{(z_1^2+z_2^2-2\rho z_1 z_2)}{2(1-\rho^2)}}\, dz_1 dz_2 . \tag{F.8}$$

Let us define

$$y_1 = \frac{z_1}{\sqrt{2(1-\rho^2)}} \qquad y_2 = \frac{z_2}{\sqrt{2(1-\rho^2)}} \quad . \tag{F.9a,b}$$

Then, Eq. (F.8) reduces to

$$\frac{\partial R}{\partial \rho} = \frac{2}{\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{y_1 y_2}\, e^{-(y_1^2+y_2^2-2\rho y_1 y_2)}\, dy_1 dy_2 \ . \tag{F.10}$$

Using the integral given on the bottom of p. 207 and the top of p. 208 of the Wax edition of Ref. 11, we have, formally, for n and m both equal to -1 and ρ equal to $-\cos\phi$,

$$\frac{\partial R}{\partial \rho} = -\frac{4}{\pi\sqrt{1-\rho^2}}\, \frac{\Gamma(1-\frac{1}{2})\ \Gamma(1-\frac{1}{2})}{(\sin\phi)^{-1}} \cos\phi\, {}_2F_1(1,1;\tfrac{3}{2};\ \cos^2\phi)$$

$$= \pm 4\rho\, {}_2F_1(1,1;\tfrac{3}{2};\ \rho^2) \quad , \tag{F.11}$$

where we have substituted $\Gamma(1/2) = \sqrt{\pi}$ and $\rho = -\cos\phi$ (hence, $\sin\phi = \pm\sqrt{1-\rho^2}$) and where ${}_2F_1(\ldots)$ is the hypergeometric function.

However, using Eqs. (A.1.35) and (A.1.39a) of Ref. 23 on pp. 1076 and 1077, we have

$$
\begin{aligned}
{}_2F_1(1,1;\ \tfrac{3}{2};\ \rho^2) &= (1-\rho^2)^{-\frac{1}{2}}\ {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};\ \tfrac{3}{2};\ \rho^2) \\
&= (1-\rho^2)^{-\frac{1}{2}}\ \frac{\arcsin\rho}{\rho} \quad . \qquad \text{(F.12)}
\end{aligned}
$$

Thus, combining Eqs. (F.11) and (F.12) yields

$$
\frac{\partial R}{\partial \rho} = \pm \frac{4}{\sqrt{1-\rho^2}} \arcsin\rho \quad . \qquad \text{(F.13)}
$$

To determine R, we may integrate Eq. (F.13) with respect to ρ:

$$
R(\tau) = \pm 4 \int_0^{\rho(\tau)} \frac{\arcsin\xi}{\sqrt{1-\xi^2}}\, d\xi + c \quad , \qquad \text{(F.14)}
$$

where c is the constant of integration and ξ is a "dummy variable." Now, when $\rho(\tau) = 0$, $z_h(t)$ and $z_h(t+\tau)$ are uncorrelated according to Eq. (F.3). In this case, it follows from Eq. (F.1) that

$$
R = \{E[\ln z_h^2]\}^2 = 4\{E[\ln z_h]\}^2 \quad , \text{ for } \rho = 0 \ , \qquad \text{(F.15)}
$$

which, according to Eq. (F.14), is our constant of integration. Thus, we have from Eqs. (F.14) and (F.15),

$$
R(\tau) = 4\left\{E^2[\ln z_h] \pm \int_0^{\rho(\tau)} \frac{\arcsin\xi}{\sqrt{1-\xi^2}}\, d\xi\right\} \qquad \text{(F.16)}
$$

However,

$$
\frac{d}{d\xi} \arcsin\xi = \frac{1}{\sqrt{1-\xi^2}} \quad ; \qquad \text{(F.17)}
$$

therefore, if we let $F(\xi) = \arcsin\xi$, the integral in Eq. (F.16) is of the form

$$\int_0^{\rho(\tau)} F(\xi)\ F'(\xi)d\xi = [F(\xi)]^2 \Bigg|_0^{\rho(\tau)} - \int_0^{\rho(\tau)} F(\xi)\ F'(\xi)d\xi \quad , \qquad (F.18)$$

where the prime denotes differentiation. Thus, from Eq. (F.18), we have

$$\int_0^{\rho(\tau)} F(\xi)\ F'(\xi)d\xi = \frac{1}{2}\ \{[F(\rho)]^2 - [F(0)]^2\} \quad . \qquad (F.19)$$

Using $F(\xi) = \arcsin\xi$ and noting that $\arcsin 0 = 0$, we have, by combining Eqs. (F.16) and (F.19),

$$R(\tau) = 4\{E^2[\ell n\ z_h] \pm \frac{1}{2} \arcsin^2 \rho(\tau)\} \quad . \qquad (F.20)$$

We may now determine the correct sign in Eq. (F.20). When $\rho = 0$, $\arcsin \rho = 0$. On the other hand, when $\rho = 1$, $\arcsin \rho = \pi/2$. However, $\rho = 1$ occurs when $\tau = 0$, and for this value of τ, $R(\tau)$ must achieve a maximum. This is possible only with the plus sign in Eq. (F.20). Consequently, the correct sign in Eq. (F.20) yields

$$R(\tau) = \{E[\ell n\ z_h^2]\}^2 + 2 \arcsin^2 \rho(\tau) \quad . \qquad (F.21)$$

The result of Eq. (F.21) can be checked as follows. At $\tau = 0$, we have $\rho(\tau) = 1$; hence, $\arcsin \rho = \pi/2$ at this point. Consequently, Eq. (F.21) gives, for this value of $\tau = 0$,

$$R(0) = \{E[\ell n\ z_h^2]\}^2 + \frac{\pi^2}{2} \quad . \qquad (F.22)$$

But, from Eq. (F.1), we have

$$R(0) = E\{[\ell n\ z_h^2]^2\} \quad . \qquad (F.23)$$

By combining Eqs. (F.22) and (F.23), we can see that Eq. (F.21) yields the result

$$E\{[\ln z_h^2]^2\} - \{E[\ln z_h^2]\}^2 = \frac{\pi^2}{2} \quad ; \tag{F.24}$$

that is, Eq. (F.21) predicts that the variance of $\ln z_h^2$ is equal to $\pi^2/2$. Since z_h^2 is, by assumption, Gaussian with zero mean and unit variance, the result of Eq. (F.24) can be checked directly. For the expected value of $\ln z_h^2$, we have, using the fact that $\ln z_h^2$ is an even function of z_h,

$$\begin{aligned} E\{\ln z_h^2\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \ln z_h^2 \, e^{-\frac{z_h^2}{2}} \, dz_h \\ &= \frac{4}{\sqrt{2\pi}} \int_0^{\infty} \ln z_h \, e^{-\frac{z_h^2}{2}} \, dz_h \\ &= -\,(C + \ln 2) \quad , \end{aligned} \tag{F.25}$$

where C is Euler's constant,

$$C = 0.577215\ldots \quad , \tag{F.26}$$

and where the last line in Eq. (F.25) was obtained using Formula (4.333) on p. 574 of Ref. 24. For the expected value of $[\ln z_h^2]^2$, we have, again using the fact that $\ln z_h^2$ is an even function of z_h,

$$\begin{aligned} E\{[\ln z_h^2]^2\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\ln z_h^2)^2 \, e^{-\frac{z_h^2}{2}} \, dz_h \\ &= \frac{8}{\sqrt{2\pi}} \int_0^{\infty} (\ln z_h)^2 \, e^{-\frac{z_h^2}{2}} \, dz_h \ . \end{aligned} \tag{F.27}$$

To put Eq. (F.27) into the form of a tabulated integral, we substitute $\xi^2 = z_h^2/2$. With this substitution, Eq. (F.27) becomes

$$E\{(\ln z_h^2)^2\} = \frac{8}{\sqrt{\pi}} \int_0^\infty [\ln (\sqrt{2\xi})]^2 \, e^{-\xi^2} \, d\xi$$

$$= \frac{8}{\sqrt{\pi}} \int_0^\infty [\ln \sqrt{2} + \ln\xi]^2 \, e^{-\xi^2} \, d\xi$$

$$= \frac{8}{\sqrt{\pi}} \int_0^\infty [(\ln \sqrt{2})^2 + 2(\ln \sqrt{2})(\ln\xi) + (\ln\xi)^2] \, e^{-\xi^2} d\xi$$

$$= I_1 + I_2 + I_3 \quad , \tag{F.28}$$

where I_1, I_2, and I_3 are the three terms that result from integrating each term within the brackets in Eq. (F.28) separately and then multiplying each by the common factor $8/\sqrt{\pi}$. Using, from Ref. 24, pp. 307 and 574, Formulas 3.321.3, 4.333, and 4.335.2 to evaluate I_1, I_2, and I_3, respectively, we find

$$I_1 = 4(\ln \sqrt{2})^2 = (\ln 2)^2 \quad , \tag{F.29}$$

$$I_2 = - 4(\ln \sqrt{2}) \, (C + \ln 4) = - 2(\ln 2) \, (C + \ln 4) \quad , \tag{F.30}$$

$$I_3 = (C + 2\ln 2)^2 + \frac{\pi^2}{2} = (C + \ln 4)^2 + \frac{\pi^2}{2} \quad . \tag{F.31}$$

Combining Eqs. (F.28) through (F.31), we have

$$E\{[\ln z_h^2]^2\} = I_1 + I_2 + I_3$$

$$= (\ln 2)^2 - 2(\ln 2) \, (C + \ln 4) + (C + \ln 4)^2 + \frac{\pi^2}{2}$$

$$= [(\ln 2) - (C + \ln 4)]^2 + \frac{\pi^2}{2}$$

$$= [(\ln 2) - C - 2(\ln 2)]^2 + \frac{\pi^2}{2}$$

$$= (C + \ln 2)^2 + \frac{\pi^2}{2} \quad . \tag{F.32}$$

Finally, combining Eqs. (F.32) and (F.25) we obtain the desired result

$$E\{[\ell n\ z_h^2]^2\} - \{E[\ell n\ z_h^2]\}^2 = \frac{\pi^2}{2} \quad , \qquad (F.33)$$

which is in perfect agreement with the result of Eq. (F.24). This completes the check of Eq. (F.24).

To obtain the final result for $R(\tau)$, we combine Eqs. (F.21) and (F.25); this gives

$$R(\tau) = (C + \ell n\ 2)^2 + 2\ \arcsin^2 \rho(\tau) \quad , \qquad (F.34)$$

which is the final result. Insofar as we are aware, the result of Eq. (F.34) is new. We remind the reader that $R(\tau)$ and $\rho(\tau)$ are defined by Eqs. (F.1) and (F.3), and that $\{z_h(t)\}$ is a stationary Gaussian process with zero mean and unit variance as indicated by Eqs. (F.2a,b).

APPENDIX G

METHOD FOR ESTIMATION OF $E\{\sigma_f^2\}$

In Eq. (6.13), a high-pass filtered version

$$w_h(t) = A' \, \sigma_f(t) \, z_h(t) \tag{G.1}$$

of the turbulence record $w(t)$ was considered, where $A' \, z_h(t)$ is the filtered version of the original component $z(t)$ of Eq. (2.3) which satisfies

$$E\{z^2\} = 1 \quad . \tag{G.2}$$

Let $H_h(f)$ denote the high-pass filter complex frequency response function used in obtaining $w_h(t)$ from $w(t)$. Then, using the approximation of Eq. (6.18), we may express the mean square value of $w_h(t)$ as

$$E\{w_h^2\} = E\{\sigma_f^2\} \int_{-\infty}^{\infty} |H_h(f)|^2 \, \Phi_z(f) df \quad , \tag{G.3}$$

where $\Phi_z(f)$ is the power spectral density of the component $z(t)$, which satisfies, according to Eq. (G.2),

$$\int_{-\infty}^{\infty} \Phi_z(f) df = 1 \quad . \tag{G.4}$$

Solving Eq. (G.3) for $E\{\sigma_f^2\}$ yields

$$E\{\sigma_f^2\} = \frac{E\{w_h^2\}}{\int_{-\infty}^{\infty} |H_h(f)|^2 \, \Phi_z(f) df} \quad . \tag{G.5}$$

According to the discussion in Sec. 6.1, $\Phi_z(f)$ may be assumed to have the appropriate von Karman form; thus, because $\Phi_z(f)$ is constrained by Eq. (G.4), it is defined by its integral scale. A method for estimating the integral scale was suggested in Sec. 6.1. Furthermore, $|H_h(f)|^2$ is a known function and

$E\{w_h^2\}$ can be measured from the filtered waveform $w_h(t)$. Therefore, Eq. (G.5) can be used to estimate $E\{\sigma_f^2\}$.

Let us now turn to justification of Eq. (6.92). Taking the *nth* moment of both sides of Eqs. (2.3) and (G.1), we obtain

$$E\{w_f^n\} = E\{\sigma_f^n\}\ E\{z^n\} \tag{G.6}$$

and

$$E\{w_h^n\} = (A')^n\ E\{\sigma_f^n\}\ E\{z_h^n\} \quad , \tag{G.7}$$

from which we obtain

$$\frac{E\{w_f^n\}}{E\{w_h^n\}} = \frac{E\{z^n\}}{(A')^n\ E\{z_h^n\}} \quad . \tag{G.8}$$

However, since $z(t)$ and $z_h(t)$ are both normally distributed with zero mean values, we have

$$E\{z^n\} = B^n\ E\{z_h^n\} \quad , \tag{G.9}$$

where B is a constant. Combining Eqs. (G.8) and (G.9), we have

$$E\{w_f^n\} = \left(\frac{B}{A'}\right)^n E\{w_h^n\} \quad . \tag{G.10}$$

However, from Eq. (2.3), we can write

$$E\{\sigma_f^2\} = \frac{E\{w_f^2\}}{\displaystyle\int_{-\infty}^{\infty} \Phi_z(f)df} \quad , \tag{G.11}$$

where the denominator in Eq. (G.11) is unity. Combining Eqs. (G.5) and (G.11) yields

$$E\{w_f^2\} = \frac{\int_{-\infty}^{\infty} \Phi_z(f)df}{\int_{-\infty}^{\infty} |H_h(f)|^2 \; \Phi_z(f)df} \, E\{w_h^2\} \quad , \tag{G.12}$$

which is of the form of Eq. (G.10) for $n = 2$. Equation (6.90) follows directly from Eqs. (G.10) and (G.12), if we define

$$K = \left[\frac{\int_{-\infty}^{\infty} \Phi_z(f)df}{\int_{-\infty}^{\infty} |H_h(f)|^2 \; \Phi_z(f)df}\right]^{\frac{1}{2}} \quad . \tag{G.13}$$

APPENDIX H

DERIVATION OF GENERAL EXPRESSION FOR MOMENTS OF $w_s(t)$ IN TERMS OF MOMENTS OF $w(t)$ and $w_f(t)$

Let w_s and w_f be independent random variables and denote their sum by w:

$$w = w_s + w_f \tag{H.1}$$

Let p_w, p_{w_s} and p_{w_f} denote the probability density functions of w, w_s, and w_f. It is well known - e.g., Ref. 9 - that p_w is the convolution of p_{w_s} and p_{w_f}; i.e.,

$$p_w(w) = \int_{-\infty}^{\infty} p_{w_s}(w-\xi)p_{w_f}(\xi)d\xi \tag{H.2a}$$

$$= \int_{-\infty}^{\infty} p_{w_s}(\xi)p_{w_f}(w-\xi)d\xi. \tag{H.2b}$$

Taking the nth moment of both sides of Eq. (H.2b), we have

$$M_w^{(n)} \triangleq \int_{-\infty}^{\infty} w^n p_w(w)dw$$

$$= \int_{-\infty}^{\infty} w^n \int_{-\infty}^{\infty} p_{w_s}(\xi)p_{w_f}(w-\xi)d\xi dw$$

$$= \int_{-\infty}^{\infty} p_{w_s}(\xi)\left[\int_{-\infty}^{\infty} w^n p_{w_f}(w-\xi)dw\right] d\xi$$

$$= \int_{-\infty}^{\infty} p_{w_s}(\xi)\left[\int_{-\infty}^{\infty} (\xi+u)^n p_{w_f}(u)du\right]d\xi$$

where we have introduced the change of variable $u = w - \xi$. Expanding $(\xi+u)^n$, we have

$$M_w^{(n)} \triangleq \int_{-\infty}^{\infty} p_{w_s}(\xi) \int_{-\infty}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} \xi^k u^{n-k} \right\} p_{w_f}(u)du \, d\xi$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[\int_{-\infty}^{\infty} \xi^k p_{w_s}(\xi) d\xi \right] \left[\int_{-\infty}^{\infty} u^{n-k} p_{w_f}(u) du \right], \tag{H.3}$$

where we have used the binomial expansion

$$(\xi+u)^n = \sum_{k=0}^{n} \binom{n}{k} \xi^k u^{n-k}, \tag{H.4}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{H.5}$$

are the binomial coefficients.

The last line in Eq. (H.3) is a relationship among the moments of p_w, p_{w_s}, and p_{w_f}; i.e.,

$$M_w^{(n)} = \sum_{k=0}^{n} \binom{n}{k} M_{w_s}^{(k)} M_{w_f}^{(n-k)}$$

$$= \sum_{k=0}^{n} \binom{n}{k} M_{w_f}^{(n-k)} M_{w_s}^{(k)} \quad . \tag{H.6}$$

Since the area under a probability density is unity, by definition, we always have

$$M_w^{(0)} = M_{w_f}^{(0)} = M_{w_s}^{(0)} = 1. \tag{H.7}$$

Using this fact, we may write out the relations of Eq. (H.6) for $n = 1,2,3,4$ as

$$M_w^{(1)} = M_{w_f}^{(1)} + M_{w_s}^{(1)}$$

$$M_w^{(2)} = M_{w_f}^{(2)} + 2M_{w_f}^{(1)}M_{w_s}^{(1)} + M_{w_s}^{(2)}$$

$$M_w^{(3)} = M_{w_f}^{(3)} + 3M_{w_f}^{(2)} M_{w_s}^{(1)} + 3M_{w_f}^{(1)}M_{w_s}^{(2)} + M_{w_s}^{(3)}$$

$$M_w^{(4)} = M_{w_f}^{(4)} + 4M_{w_f}^{(3)}M_{w_s}^{(1)} + 6M_{w_f}^{(2)}M_{w_s}^{(2)} + 4M_{w_f}^{(1)}M_{w_s}^{(3)} + M_{w_s}^{(4)}. \quad \text{(H.8)}$$

The above relationships may be solved successively for $M_{w_s}^{(1)}$, $M_{w_s}^{(2)}$, $M_{w_s}^{(3)}$ ··· to yield the set of relationships given by Eqs. (6.89) in the main text. The general form of these relationships is easily seen to be given by Eq. (6.87) i.e.,

$$M_{w_s}^{(n)} = M_w^{(n)} - \sum_{k=0}^{n-1} \binom{n}{k} M_{w_f}^{(n-k)}M_{w_s}^{(k)}, \; n = 1,2,3,\cdots. \quad \text{(H.9)}$$

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